Volatilities That Change with Time: The Temporal Behavior of the Distribution of Stock-Market Prices

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Abstract

While the use of volatilities is pervasive throughout finance, our ability to determine the instantaneous volatility of stocks is nascent. Here, we present a method for measuring the temporal behavior of stocks, and show that stock prices for 24 DJIA stocks follow a stochastic process that describes an efficiently priced stock while using a volatility that changes deterministically with time. We find that the often observed, abnormally large kurtoses are due to temporal variations in the volatility. Our method can resolve changes in volatility and drift of the stocks as fast as a single day using daily close prices.

keywords: spectral analysis, noise reduction, Rademacher distribution

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I. INTRODUCTION

In this paper, we study the temporal behavior of the distribution of stock prices for 24 stocks in the Dow Jones Industrial Average (DJIA). This is done using a new method of measuring changes in the volatility and drifts of stocks with time. When this method is applied to time-series constructed from the daily close of stocks, changes as fast as one day can be seen in both. Given that it is not possible to accurately *measure* (as oppose to *predict*) intraday changes in the volatility using only daily-close data, for two of the 24 stocks we have been able to reach the maximum resolution (known as the Nyquist criteria) of one day in the rate that the volatility can change, while for the great majority of the remaining stocks, we have come within one day of this maximum. We believe that this method can measure changes in the volatility and drift that occur during the trading day as well if intraday price data is used. But even with only daily-close data, we have been extraordinarily successful at determining the temporal behavior of stocks in general, and of the volatility in particular, and in the process, we have furthered our understanding of the behavior of stock prices as a whole.

We find that the stock prices of these 24 stocks can be well described by a stochastic process for which the volatility changes *deterministically* with time. On the one hand, this is a process where the yield at any one time is not correlated with the yield at any other time; the process thus describes an efficiently priced stock. On the other hand, this is a process where the predicted kurtosis agrees with the sample kurtosis of the stock; the process thus also provides a solution to the long standing problem of explaining how an efficiently priced stock can have a kurtosis that is so different from what is expected for a Gaussian distribution. Indeed, we find that abnormally large kurtoses are due solely to changes in the volatility of the stock with time. When this temporal behavior is accounted for in the daily yield, the kurtosis reduces dramatically in value, and now agrees well with model predictions. This finding is in agreement with Rosenberg's (1972) observation that the kurtosis for nonstationary random variables is larger than than the kurtosis of individual random variables. We have also determined changes in the volatility of these stocks, and for three of the 24 stocks, variations of as fast as one day can be seen. For another 16 stocks, this temporal resolution was two days or less, and for only five of the 24 stocks is this resolution longer than 2.5 days.

The behavior of the drifts for all 24 stocks can also be determined using this method, and with the same resolution as their volatility. We find that the drift for the majority of the stocks is positive; these drifts thus tend to augment the increase of the stock price caused by the random-walk nature of the stochastic process. This finding is not surprising, nor is it surprising that we find that the drift is much smaller than the volatility for all 24 stocks. What is surprising is that for three of the 24 stocks the drift is uniformly *negative*. For these stocks, the drift tends not to increase the stock price, but to depress it. That the stock price for these three stocks increase at all is because this drift is much smaller in the magnitude than the volatility. Over the short term, growth in the prices of these stocks—as they are for all 24 stocks—is due to a random walk, and thus driven more by the volatility than the drift. Indeed, this is the only reason that the prices of these stocks increase with time.

Finally, the distribution of the stock prices for the 24 DJIA stocks has been determined. When the temporal variation in the volatility is corrected for in the daily yield, we find that the resultant distribution for all but four of the stocks is described by a Rademacher distribution with the probability that the yield increases on any one day being 1/2. For the four other stocks, the distribution is described by a generalized Rademacher distribution with the probability that the yield increases on any one day being slightly greater than the probability that it decreases.

II. BACKGROUND, PREVIOUS WORK, AND A SUMMARY OF THE APPROACH

In 2005, 403.8 billion shares were traded on the New York Stock Exchange (NYSE) with a total value of $14.1 trillion dollars (see NYSE). During the same period, 468 million contracts were written on the Chicago Board Options Exchange (CBOE) with a total notional value of $12 trillion dollars. At the NYSE, traders, investors, and speculators—big and small—place bets on the movement of stock prices, whether up or down. Profits are made, or losses are reconciled, based on the changing price of the stock. As such, great effort is made to predict the movements of stock prices in the future, and thus much attention—with attending analysis—is focused on the price of stocks.

In the CBOE, traders, investors, and speculators write or enter into contacts to purchase or sell a predetermined amount of stocks at a set time in the future. Profits here are made,

or losses reconciled, based on the degree of risk that the movement of the stock will be down when expected to be up, or up when expected to be down. Here, it is not so much the price of the stock that matters. It is the amount of volatility in the stock, and predicting how stock prices may move in the future is much less important. Indeed, the pricing of options—through the Black-Scholes equation and its variants—is based on the argument that it is *not* possible to predict how the price of stocks will change in the future. In this pricing, it is taken for granted that the markets are efficient, and that earning returns which are in excess of the risk-free interest rate is not possible. All is random, and the increase in stock prices seen is due to a simple random walk with a (small) drift. Great interest is thus paid in modeling the *distribution* of stock prices, and the application of these models to the pricing of options and derivatives.

Given the $26.1 trillion dollars in trades and contracts in the NYSE and CBOE in 2005, it is not surprising that much effort has been expended in determining the properties of the stock market. Given the precipitous drop in stock market prices in October of 2008—*which occurred over period of days*—accurate determination of how these properties change with time has become even more important. Since the work by Bachelier (1900) at the turn of the 20th century, a great deal of these efforts have been focused on determining the distribution of the daily yields of stock prices (Osborne 1959a and Osborne 1959b). Inherent in this determination is determining the volatility of the distribution. Use of this volatility is now pervasive in modern finance, and is a critical ingredient in such endeavors as the pricing of options, the general assessment of risk and the determination the value of assets at risk, and the construction of optimal portfolios. That this effort continues today is indicative of the difficulty in determining this distribution, its importance in modern finance, and the financial impact that its determination can have.

While Bachelier (1990) characterized the distribution as a random walk with the prices of the stock having a given drift and a constant volatility, it has been known since the detail analysis of the behavior of stock prices by Fama (1965) that the distribution of daily yields is only approximately Gaussian; the distribution calculated by Fama—which does not take into account variations in the volatility with time—has a fatter tail than expected for a Gaussian distribution. Indeed, it is typically found that the kurtosis can be as high as 100, while by comparison the kurtosis of a Gaussian distribution is only three. This discrepancy between the distribution of daily yields as they are traditionally calculated

and the Gaussian distribution, while seemingly an inconsequential detail, nonetheless has wide-ranging consequences.

Mathematics tells us that if the distribution of daily yields of a stock is a Gaussian distribution, then the daily yield on any one day cannot depend on the daily yield on any other. This is the Central Limit Theorem (CLT), and it is embodied in a number of ways—the various forms of the Efficient Market Hypothesis (EMH) (see Fama 1970 and Fama and French 1988), and the no-arbitrage condition—in modern finance. This lack of predictability is one of the underlying assumptions used in the pricing of derivatives. Mathematics says we can also turn the statement of the CLT around, however. Namely, if the daily yield on any one day does not depend on the daily yield on any other, then the distribution of daily yields must *necessarily* be a Gaussian distribution as long as the number of days used in its determination is large enough, and as long as the distribution is well behaved.

In the face of this mathematical result, there are two possibilities. The first possibility is that the distribution of daily yields for stocks is not Gaussian. The daily yield on one day does depend on the daily yield on some other day, and it is possible, in principle, to predict future stock prices by looking at historical prices. The second possibility is that the EMH nevertheless holds, and there are good, albeit unknown, reasons for the unexpectedly large kurtosis. The situation is further muddied when the autocorrelations of the daily yield of stocks are calculated. It is well known from these calculations that the value of the daily yield on different days are uncorrelated with each other, and we have seen this behavior for the stocks studied here as well. This independence extends also to other asset classes, as shown by Kendall (1953).

There have been numerous attempts at using other distributions—the Levy and its generalization, the Pareto, proposed by Mandelbrot (1963), the Student t-Distribution proposed by Blattberg and Gonedes (1974), and the discrete mixture of Gaussian distributions model proposed by Kon (1984)—to describe the distribution of stock prices (see Töyli, Sysi-Aho, and Kaski 2004 for an overview and assessment). These attempts are based on the belief that the second of the two possibilities holds, and that the reason for the overly large kurtosis is because the distribution used to describe the stock was not correct. As such, for these distributions the daily yield on any one day also does not depend on the daily yield on any other day, and the consequences of the CLT is instead evaded in various ways. These

approaches have had various degrees of success. For example, while the Pareto distribution does have a fatter tail than the Gaussian distribution and has a kurtosis that can agree with observations, all moments with order greater than an integer k—which determines the power-law behavior of the distribution—is ill defined; in this way, the distribution is not well behaved, and thus does not fall within the class of distributions for which the CLT is applicable. For the Levy distribution, the volatility itself (as well as all higher moments) is ill-defined, requiring the truncation of the distribution to price options using this model, as described in Kleinert (2002). The Student-t Distribution differs significantly from the Gaussian distribution only when the number of data points are small (thereby evading the CLT), which begs the question of what happens when this distribution is applied to time-series with more than, say, 200 terms in it. Kon's model is a of mixture of Gaussian distributions, and thus the moments of his distribution are all finite. However, while the model is effective at describing the large kurtosis of stocks, it is nonetheless an empirical model; the origin of the discrete mixture is not known, and the parameters used in its construction are determined only after the model is fitted to the stock data.

Our approach is also based on the belief that the second of the two possibilities hold. But unlike the previous attempts at describing the distribution of stock prices mentioned above, we find that the underlying reason for the overly large kurtosis is because time variations in the distribution of stocks have not been properly taken account of. As observed by Rosenberg (1972), it is often assumed that the distribution of stock prices being analyzed does not change during the period of interest. This assumption was certainly made for all the models described above. In contrast to these approaches, we will take time variations in the distribution explicitly into account. Doing so results in a distribution that can both explain the abnormally kurtosis, and still have the property that the yield on any one day is not correlated with the yield on any other. In the process, we will also be able to determine, for the first time, how the volatility and the drift changes instantaneously with time.

That the volatility of stocks changes with time is not a new observation. This behavior has been known since at least the work by Osborne (1962) (see also Lo 1988), and analyzed explicitly by Rosenberg (1972). Indications of this have been reported by many others since then (Ball and Torous 1985, French and Roll 1986, Conrad and Kaul 1988, Andersen and Bollerslev 1997, Kullmann, Töyli, Kertesz, Kanto, and Kaski 1999, Nawroth and Peinke 2006). Much effort has since been made to determine how this volatility—and thus necessar-

ily how the distribution—changes with time, with the main focus of this effort on extending the usual random walk description of stock market prices. This has lead to the introduction of the jump-diffusion model proposed by Merton (1976), where discrete, random jumps in the prices of a stock in time are incorporated in continuous stochastic processes, and to stochastic volatility models developed over a number of years by Praetz (1972), Christie (1982), Hull and White (1987), Scott (1987) and Heston (1993) where the volatility itself is modeled as a stochastic process with its own drift and volatility (see also Muzy, Delour, and Bacry 2000 for a multifractal-inspired, stochastic volatility model). However, as it was pointed out by Hull and White (1987), methods for directly *measuring* time-varying volatilities were not, at the time, known.

This inability to directly measure variations in the volatility has greatly constrained efforts in studying how the volatility of real market data varies with time. To a great extent, it has also driven the development of stochastic volatility models. By characterizing the volatility as a stochastic process, a time varying volatility can be modeled using a comparatively simple choice of a constant drift and a constant volatility for the process. Even then, however, parameters in stochastic volatility models are determined not by a direct analysis of the daily yields of stock prices, but are instead determined indirectly. Namely, the price of an option for a stock is calculated for the process in terms of a set of model parameters, and these parameters are then set by adjusting their values until the calculated price agrees with the market price of the option.

The inherent difficulty in determining from market data how a distribution changes with time is described in Boyle and Anathanarayanan (1977), and is straightforward to understand. To determine the distribution of a stock, a collection of stock prices is required; the larger the collection, the better. Since stock prices change sequentially in time, this collection has to be done over a period of time, and because a relatively large collection is needed, this period must be correspondingly long. For example, most distributions are calculated using the daily close of a stock, if for no other reason then because these prices are readily available in the public domain. If the collection of prices used is as large as 500 daily closes, then the stock prices in this collection must span a period of nearly two years; Fama (1965), for example, used stock prices that span a period of up to six years in his analysis. Using a collection of 500 stock prices to determine the distribution of the stock through standard methods means that one is tacitly assuming that the price of the stock

two years ago belongs to the same distribution—*with the same volatility*—as the price of the stock today. This strains credibility, especially given the rapid movements in the markets during the last quarter of 2008. While it is possible to calculate the distribution with a fewer number of stock prices and thus shorten the period of time over which they are collected, statistical errors inherent in determining the distribution are proportional to $1/\sqrt{N}$, where N are the number of data points in the sample, and will thus be correspondingly larger. At one point, the period of time would be so short that we would not be able to say whether distribution is Gaussian or not. Indeed, we only have to look at the extreme case where the period is so short that there are only three stock prices collected over three days, resulting in only two daily yields to determine the whole distribution; this clearly cannot be done with any certainty whatsoever! (This inherent difficulty has lead to the development of other approaches to calculating volatility such as those found in Ball and Torous 1984, Parkinson 1980, and Longin 2005) where the number of daily close needed is reduced.)

Mathematics does not require that the distribution remains constant. The general theory of stochastic processes allows for volatilities that change with time. In fact, we will show below that even though the volatility of a Gaussian distribution may change with time, the daily yield on any one day need not depend on the daily yield on any previous day; the EMH still holds for this case. Instead, what has been lacking up to now is a method for *calculating* the statistical properties of a stock when the volatility changes with time. This we have been able to do.

Our approach is based on the observation that when the volatility depends solely on time, we can remove the time dependence of the distribution by dividing the daily yield by the volatility. This standardizes the daily yield, and a Gaussian distribution with a time-varying volatility is mapped into a Gaussian distribution with unit volatility. The volatility of this distribution is clearly constant, and thus the standardized daily yields all belong to the *same* distribution. The inherent difficulty in determining a distribution that changes with time mentioned above is thus circumvented. Indeed, large collections of stock prices are now a benefit—they result in smaller standard errors—and not a detriment. That the volatility of the mapped distribution is *known* then allows us to determine how the volatility of the original daily yield changes with time. In addition, it is readily apparent from our analysis that the distribution of standardized yields is equivalent to a special case of the binomial distribution, and this observation allows us to extract easily the temporal behavior of the

drift of the yield as well.

This approach is straightforward, and at its heart resembles the process one goes through in using a table of values for the cumulative standardized Gaussian distribution: The random variable at hand is scaled with its volatility to get the standardized Gaussian distribution with unit volatility. The difference is that in our case the volatility is not know a priorí; it must be determined. This is done using a combination of statistical methods, Fourier analysis, and signal processing techniques. While prevalent in other fields, many of these techniques are not commonly found in the finance or business literature, and it would be easy to become too involved with the mathematics while neglecting the finance when presenting our results. To avoid doing so, we will focus on finance in the main body of the paper, and when our model of stock prices is constructed, it will be motivated by, and justified with, an analysis of the time-series of the stocks at hand. Importantly, a validation of each step taken will made. Only enough of the underlying mathematical analysis needed to explain the essential ideas behind our approach will be presented in the main body of the paper; we will refer the reader to the appendices for many of the details. Our analysis will be applied explicitly to Coca Cola stock in this paper to demonstrate the underlying ideas behind the approach. This stock is chosen out of the 24 because for our purposes its underlying behavior is representative of all the others. Analysis of the other 23 stocks studied here follow in much the same way, and we will only present a summary of the results for them, along with graphs of the volatilities for all 24 stocks as a function of trading day.

III. MODEL VALIDATION AND OUR CHOICE OF STOCKS

It would not be an exaggeration to say that the only characterizations of a stock that is not model dependent is the price per share that it was sold at, the day and time it was sold, and the total number of shares of the stock that was sold over a given period of time. These are the only characterizations that are objective and verifiable, and for whom all can agree on how they are obtained. The distribution of the daily yields of a stock certainly is not, and herein lies the problem: How should any model of stock prices be validated?

To see how difficult the problem of validation is to resolve (this issue was explicitly studied by Magdon-Ismail and Abu-Mostafa 1998 for volatility models), consider the volatility of the 24 stocks considered here. As we will calculate the volatility of these stocks, it would

seem that a comparison of the volatilities we obtain here with the volatilities calculated using any one of the many other approaches in the literature would be an effective way of assessing the validity of our model. However, irrespective of the approach taken to make this calculation, assumptions about the behavior of the stock will have already been made. The historical volatility, for example, uses a moving average to calculate the volatility on any given trading day. It implicitly assumes that the volatility does not change significantly over the window of time used when calculating the average, and thus cannot effectively measure changes in the volatility that occur within this window. The implicit volatility, developed over a series of papers by Latané and Rendleman (1976), Schmalensee and Trippi (1978), and Beckers (1981), can measure instantaneous changes in the volatility, but it is calculated by inverting the Black-Scholes (or any other) equation for pricing options, and thus implicitly assumes that the particular pricing equation used accurately prices the option at any given time. Autoregression approaches to calculating the volatility—such as the exponentially weighted moving average (EWMA), the autoregregressive conditional heteroskedasticity (ARCH) proposed by Engle (1982), the generalized autoregregressive conditional heteroskedasticity (GARCH) proposed by Bollerslev (1986), and a new approach that combines autoregressive and Fourier (spectral) analysis techniques proposed by Bollerslev and Wright (2001)—are designed more to *manage* volatilities that change with time than to *characterize* them. They depend on one or more parameters that must subsequently be set using some property of a stock, and are not designed to *determine* how the volatility changes. Stochastic volatility models explicitly consider volatilities that change (randomly) with time, but to determine how this volatility changes, the approach adjusts the parameters that determine the volatility until the predicted option prices agree with market prices (see Lamoureux and Lastrapes 1993 for a test of this approach). Using a comparison of volatilities to validate models is therefore more a comparison of the underlying models of the market or methods of calculation than it is of the volatilities themselves. Indeed, the question of which approach to calculating the volatility is the better one is one that has been address many times over the years (see Day and Lewis 1992, Canina and Figlewski 1993, Jorion 1995, Figlewski 1997, Andersen and Bollerslev 1998, Chong, Ahmad, and Abdullah 1999, Szakmary, Ors, Kim, and Davidson II 2003, and McMillan and Speight 2004), apparently without consensus.

This difficulty in validating models is particularly inopportune here. While many of the

techniques we have used in this paper has been long used in other fields, our approach in this paper is novel, and have not been used in the analysis of stock market prices before. We therefore take a particularly stringent approach to validating our model. First, the model must be able to explain the observed properties of the stocks. This we accomplish by construction. Properties of the stock price are presented *first*, and the model is then constructed explicitly to describe them. Second, the model must be self-consistent, and must be able to predict some property the stock price, which can subsequently be verified. All models of stock market prices make a certain set of underlying assumptions about properties of the price; these assumptions have consequences. These consequences can in turn be used to predict properties of the stock price that can then be used to validate it. For our model, the distribution of standardized daily yields is described by a Rademacher distribution or its generalization. This distribution gives specific values for the population skewness and kurtosis, and they provide a simple and statistically meaningful approach to validating our model. Specifically, we calculate the sample skewness and kurtosis from each stock's time-series. We then compare this sample skewness and kurtosis to the population skewness and kurtosis predicted by our model. If the sample skewness and kurtosis agree with the population skewness and kurtosis of our model at the 95% confidence level (CL), we assert that our model is valid. In fact, we find that this agreement holds for all 24 stocks considered here, and it does so over the whole of the time period spanned by their time series. Indeed, for a number of the stocks, *this period spans over 80 years.*

It is because of this operational approach to validating our model that we chose to analyzed stocks from the DJIA. First, all the stocks in the DJIA are large caps, and have a large daily trading volume; they are precisely the type of stocks for which we expect the market to be efficient. They are in this way similar, and we would expect they can be described by the same type of model. Second, each of these companies has been publicly traded for a number of years. We therefore have access to a large collection of daily close prices for these stocks with which to construct their time-series. These time series, for example, range in time from as short as 5,090 trading days for Citigroup, to as long as 21,527 trading days for Exxon-Mobil. The availability of a large sample of daily close is particularly important as we will be numerically assessing the validity of each step in the construction of our model. With such large collections of stock prices, standard errors in our calculation can be as small as 0.7%, and as such, we are able to say with a great deal

of certainty whether or not our approach is self-consistent. Third, the 24 chosen were the simplest, in terms of their ownership, of the 30 stocks listed in the DJIA. The six DJIA not chosen were recently involved in mergers or acquisitions, which introduces unwanted complications; an assessment of the temporal of these stock prices may not be as clear cut as the 24 stocks considered here.

A detailed description of how the time-series are constructed is given in Appendix A, where any particularities in the analysis of stocks are listed as well. A list of these stocks given in terms of their stock symbol is presented in Table I along with the starting date of the time-series and the total number of daily yields in each. The ending date for all 24 time-series is December 29, 2006.

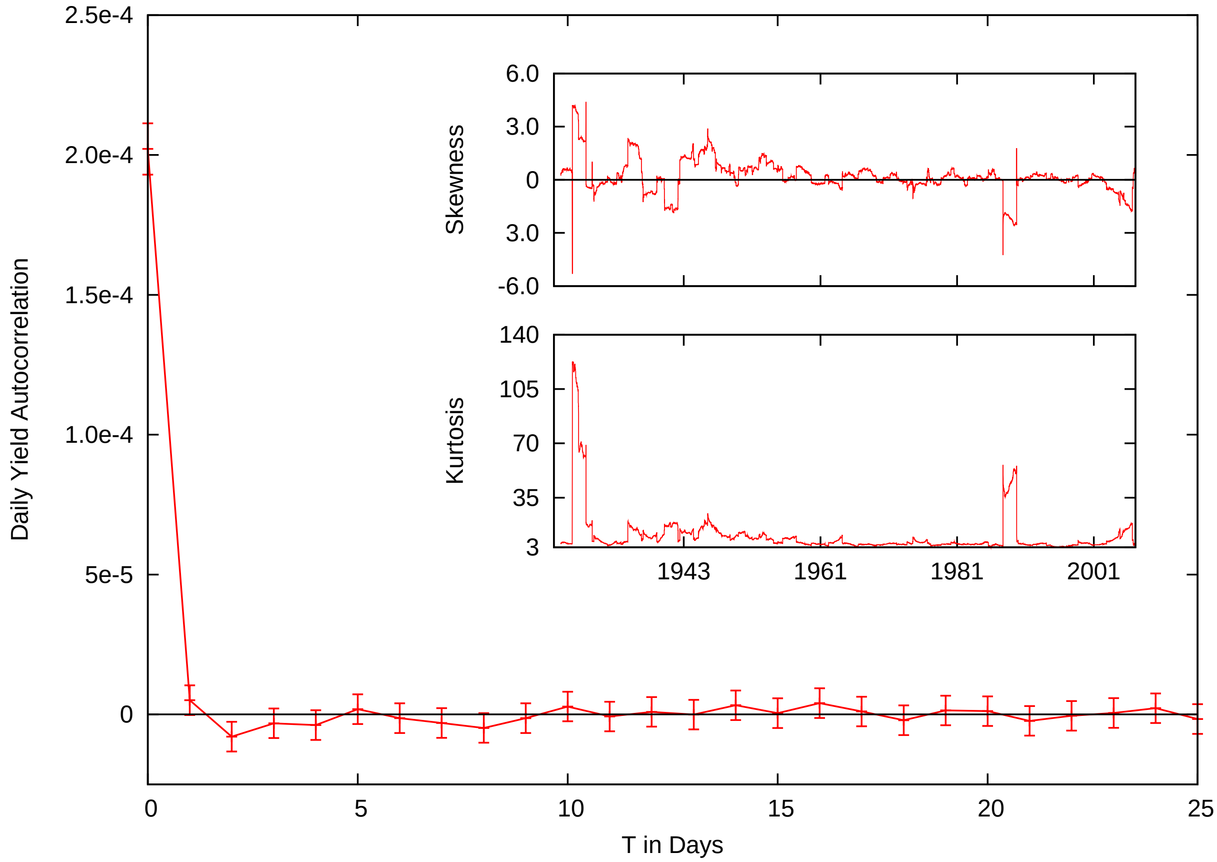
IV. A TEMPORAL MODEL OF STOCK MARKET PRICES

We begin our study of the temporal behavior of stock market prices with an application to finance. Specifically, for the 24 stocks considered here we study whether the daily yield on December 29, 2006 depends on the daily yield on any day previous to it. This property of the market, which has direct implications in finance, will be used as the starting point for the construction of our model of stock prices.

A. An Inherent Contradiction

Shown in Fig. 1 is a graph of the autocorrelation function of the daily yield for Coca Cola using Eq. (B6) from the Appendix B. This autocorrelation is calculated between the daily yield of the stock on December 29, 2006, and the daily yield T days *before* the 29th. The graph thus shows the dependency of the yield on the 29th on the yield on any previous day. If the yield on the 29th depends on the yield on day T, then the autocorrelation function will not vanish on that day at the 95% CL. If, on the other hand, the yield on the 29th does not depend on the yield on day T, then the autocorrelation function will be within statistical error of zero.

Also shown on the graph in Fig. 1 is the errorbar for each of the calculated values of the autocorrelation function. These errorbars are set at the 95% CL, which is 1.96 times the standard error calculated using Eq. (B11) for the autocorrelation function on that day. They



Indications of an Efficiently Priced Stock

FIG. 1: The autocorrelation of the daily yield for Coca Cola is shown in the main figure, with the time, T, labeling the number of trading days *before* December 29, 2006. Also included at each data point are errorbars set at ± 1.96 times the standard error. In the insert, graphs of the sample skewness and the kurtosis of the stock calculated using a 251-day moving average are shown.

thus set the 95% confidence interval (CI) about the calculated value for the autocorrelation function. If the value of the autocorrelation function falls within its errorbar of zero, there is a 95% probability that the autocorrelation on this day equals zero. With 21,522 total trading days in the time-series for Coca Cola, the standard error for the values of the autocorrelation function shown in the graph is roughly 0.7%, and is thus quite small; the errorbars shown are correspondingly small. The standard error for the majority of the stocks studied here are equally small.

All but one of the errorbars for the autocorrelation shown in Fig. 1 straddles the x-axis. As such, we can say that the value of the autocorrelation function for $T > 0$ is within a 95%

CL of zero for all but one day. Indeed, when we continue this calculation all the way back to December 31, 1925, the starting date for the time-series, we find that the autocorrelation function for the daily yield on 20,279 out of a total of 21,522 trading days fall within the 95% CI of zero; the autocorrelation function on 1,243 trading days, or 6% of the trading days, fall outside of the 95% CI (see Table I). This does not necessarily mean that there is a correlation between the 29th and these 1,243 trading days, however. Statistically, we would expect values of the autocorrelation function to exceed the 95% CI on 5%, or 1,076, of the trading days. We can only conclude that on at least 1%, or 215, of the trading days the autocorrelation function does not vanish for $T > 0$. If instead a 99% CI in chosen, we find that the value of the autocorrelation function falls within the 99% CI of zero for 21,172 out of 21,522 trading days; they fall outside of the 99% CI on only 2%, or 350, of the trading days. We can therefore still conclude that for at least 1%, or 215, of the trading days the autocorrelation function may not vanish for $T > 0$.

The autocorrelation function of the daily yield for all 24 stocks have been calculated for the length their time-series, and we have found that the autocorrelation function for these stocks behave similarly to Coca Cola's. Namely, the autocorrelation function is maximum at $T = 0$, and it does not vanish for at least 1% to 3% of the trading days for each stock; for Citigroup and Verizon, this percentage is even lower. We may conclude from this analysis that for the vast majority of the time the daily yield of these stocks on any one day is not correlated with the daily yield on any subsequent day; the market is thus extremely efficient for these 24 stocks. In addition, we will show below that for the 1% to 3% of the trading days when the autocorrelation function does not vanish, this is due to changes in the volatility of the stock with time, and not to correlations between daily yields.

Based on the above analysis, it would seem that the usual stochastic process with a constant volatility would be a good model for these stocks. The lack of dependence of the daily yield on any one day from any other is precisely the property inherent in such a model. There are, however, other properties of the distribution of daily yields for stocks that any model would have to explain as well, and it is here that the constant-volatility model of stocks is lacking.

Shown in the insert of Fig. 1 is the sample skewness of the daily yields for Coca Cola calculated using a 251-day moving average. If indeed the stock price of the stock is well described by a stochastic process with a drift and a constant volatility, then we would expect

the skewness of the daily yield to be zero. For Coca Cola, we find that the skewness ranges from -5.3 ± 6.5 to 4.4 ± 5.3. Although the skewness is large in magnitude, its standard error is correspondingly large, and we find that the skewness exceeds the 95% CI of zero on only 943 out of 21,022 days, or 4%, of the time. Thus, the sample skewness calculated using a 251-day moving average agrees with what is expected from modeling the yield of the stock using a stochastic process with constant volatility.

The situation is quite different for the kurtosis, however. Shown also in the insert of Fig. 1 is the sample kurtosis of the daily yields for Coca Cola calculated with the same 251-day moving average. Although the kurtosis for a daily yield described by a stochastic process with a constant volatility is expected to be three, what we find instead is that the sample kurtosis calculated for the Coca Cola time-series ranges in value from 2.92 ± 0.22 to 122 ± 49. Like the skewness, the standard error for the kurtosis is large when the kurtosis is large, but unlike the skewness, the error is not overwhelmingly large. We find that the kurtosis exceeded the 95% CI of three on 15,393 out of 21,022 days, or 72%, of the time. For the great majority of the trading days in the time-series, the kurtosis is different from that expected for a stochastic process with constant volatility.

We have done this calculation for all 24 stocks, and these results are not unique to Coca Cola. This, then, is the contradiction inherent in using a stochastic process with constant volatility to model stock market prices. On the one hand, calculations of the autocorrelation function indicate that the market is extremely efficient for these stocks, which is consistent with a stochastic process with constant volatility. On the other hand, calculation of the kurtosis for these stocks are much larger than expected for such a process. We will use this contradiction to guide the construction of our model in the analysis below.

TABLE I: Autocorrelations for the 24 DJIA Stocks

	Starting		Daily Yield		Standardized Daily Yield	
	Date	N_T	$> 95\%$ CI	$> 99\%$ CI	$> 95\%$ CI	$> 99\%$ CI
C	10/29/86	5088	227 (4%)	53 (1%)	256 (5%)	45 (1%)
MSFT	03/13/86	5248	221 (4%)	63 (1%)	292 (6%)	60 (1%)
VZ	02/16/84	5770	236 (4%)	58 (1%)	306 (5%)	57 (1%)

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	Starting		Daily Yield		Standardized Daily Yield	
	Date	N_T	> 95% CI	> 99% CI	> 95% CI	> 99% CI
INTC	12/14/72	8592	516 (6%)	159 (2%)	466 (5%)	93 (1%)
AXP	12/14/72	8592	412 (5%)	101 (1%)	391 (5%)	78 (1%)
AIG	12/14/72	8592	470 (5%)	137 (2%)	470 (5%)	96 (1%)
WMT	11/20/72	8608	458 (5%)	118 (1%)	415 (5%)	85 (1%)
HPQ	03/03/61	11524	610 (5%)	169 (1%)	575 (5%)	112 (1%)
DIS	11/12/57	12386	741 (6%)	182 (1%)	624 (5%)	135 (1%)
AA	06/11/55	14014	688 (5%)	153 (1%)	710 (5%)	158 (1%)
MRK	05/15/46	15444	946 (6%)	233 (2%)	835 (5%)	190 (1%)
MMM	01/15/46	15544	733 (5%)	253 (2%)	724 (5%)	145 (1%)
JNJ	09/25/44	15920	759 (5%)	163 (1%)	734 (5%)	145 (1%)
PFE	01/17/44	16126	812 (5%)	181 (1%)	812 (5%)	181 (1%)
BA	09/04/34	18946	1281 (7%)	360 (2%)	980 (5%)	200 (1%)
CAT	12/02/29	20358	1691 (8%)	702 (3%)	1027 (5%)	239 (1%)
PG	08/12/29	20442	1644 (8%)	678 (3%)	1026 (5%)	223 (1%)
GE	12/31/25	21518	1474 (7%)	545 (3%)	1104 (5%)	242 (1%)
GM	12/31/25	21518	1370 (6%)	488 (2%)	1120 (5%)	251 (1%)
DD	12/31/25	21520	1520 (7%)	532 (2%)	1071 (5%)	215 (1%)
MO	12/31/25	21520	1804 (8%)	762 (4%)	1047 (5%)	242 (1%)
IBM	12/31/25	21522	1387 (6%)	461 (2%)	1095 (5%)	193 (1%)
KO	12/31/25	21522	1243 (6%)	350 (2%)	1124 (5%)	230 (1%)
XOM	12/31/25	21526	1498 (7%)	477 (2%)	1096 (5%)	220 (1%)

B. The Continuous Model

In this section, we show that for continuous stochastic models of stock prices with a *deterministic* volatility that changes with time, the yield at time, t, does not depend on the

yield at any other time, t'. Such a stock price is thus able to model the properties of the autocorrelation function found for the 24 stocks above. In a later section, we will show that the time-variation in the volatility can also explain the abnormally large sample kurtosis.

Take as the price of the stock at any time, t, the continuous function $S(t)$. This is an approximation, of course. Stocks are bought and sold in discrete time periods, and the prices of these transactions are always recorded in discrete units. It is, however, easier to develop an understanding of the model, and to show a number of properties of it, using this continuous approximation instead of using a discrete time-series of stock prices. In the next section, when we develop a recursion relation for the volatility, we will consider real-world data, and will discretize the continuous model presented here.

Our model for $S(t)$ is a stochastic process with a drift, $\widetilde{\mu}(t)$, and a volatility, $\sigma(t)$, that change only with time:

$$\frac{1}{S}\frac{dS}{dt} = \widetilde{\mu}(t) + \sigma(t)\xi(t), \tag{1}$$

where $\xi(t)$ is a Gaussian random variable such that

$$E[\xi(t)] = 0, \qquad \text{and} \qquad E[\xi(t)\xi(t')] = \delta(t - t'). \tag{2}$$

Here, $E[\xi]$ is the expectation value of ξ over a Gaussian distribution, and $\delta(t)$ is the Dirac delta function. We emphasize that while Eq. (1) may have a form that is similar to various stochastic volatility models of the stock market, for us $\sigma(t)$ is a *deterministic* function of time; it does not have the random component that is inherent in stochastic volatility models.

As usual, it is more convenient to work with $u(t) \equiv \ln[S(t)]$; for continuous compounding, du/dt is then the instantaneous yield of the stock. In terms of $u(t)$, Eq. (1) reduces to

$$\frac{du}{dt} = \mu(t) + \sigma(t)\xi(t), \tag{3}$$

where $\mu(t) = \widetilde{\mu}(t) - \sigma^2(t)/2$.

It is straightforward to show that for this stochastic process the instantaneous yield at time, t, does not depend on the yield at any other time, t'. To do so, consider the expectation value

$$E\left[\left(\left.\frac{du}{dt}\right|_t - \mu(t)\right)\left(\left.\frac{du}{dt}\right|_{t'} - \mu(t')\right)\right] = E[\sigma(t)\sigma(t')\xi(t)\xi(t')]. \tag{4}$$

Because $\sigma(t)$ is a deterministic function, it can be moved outside the expectation value so

that $E[\sigma(t)\sigma(t')\xi(t)\xi(t')] = \sigma(t)\sigma(t')E[\xi(t)\xi(t')]$. Using Eq. (2), we then conclude that

$$E\left[\left(\frac{du}{dt}\bigg|_t - \mu(t)\right)\left(\frac{du}{dt}\bigg|_{t'} - \mu(t')\right)\right] = \sigma(t)^2\delta(t-t'), \tag{5}$$

so that the autocorrelation function of the instantaneous yield vanishes unless $t = t'$; the yield of the stock at any one time does not depend on the yield at any other time. Our model thus describes a market for the stock that is efficient. This is to be expected. At each instant, t, Eq. (3) describes a Gaussian distribution with drift, $\mu(t)$ and volatility, $\sigma(t)$, and it is well known that for a Gaussian distribution the daily yield on any one day is not correlated with the daily yield on any other.

Note that if the volatility was a function of u as well as t, or if it was itself a stochastic process, as it is taken to be in stochastic volatility models, we could not have moved the volatilities outside the expectation value to obtain Eq. (5). In these cases, it is not clear whether the yield of the stock at any one time depends on the yield at any other time.

Formally, the solution to Eq. (3) is straightforward. If $\sigma(t) > 0$ for all t, divide through by $\sigma(t)$, and then reparametize time by taking

$$\tau = \int_0^t \sigma(s)ds. \tag{6}$$

Equation (3) then simplifies to

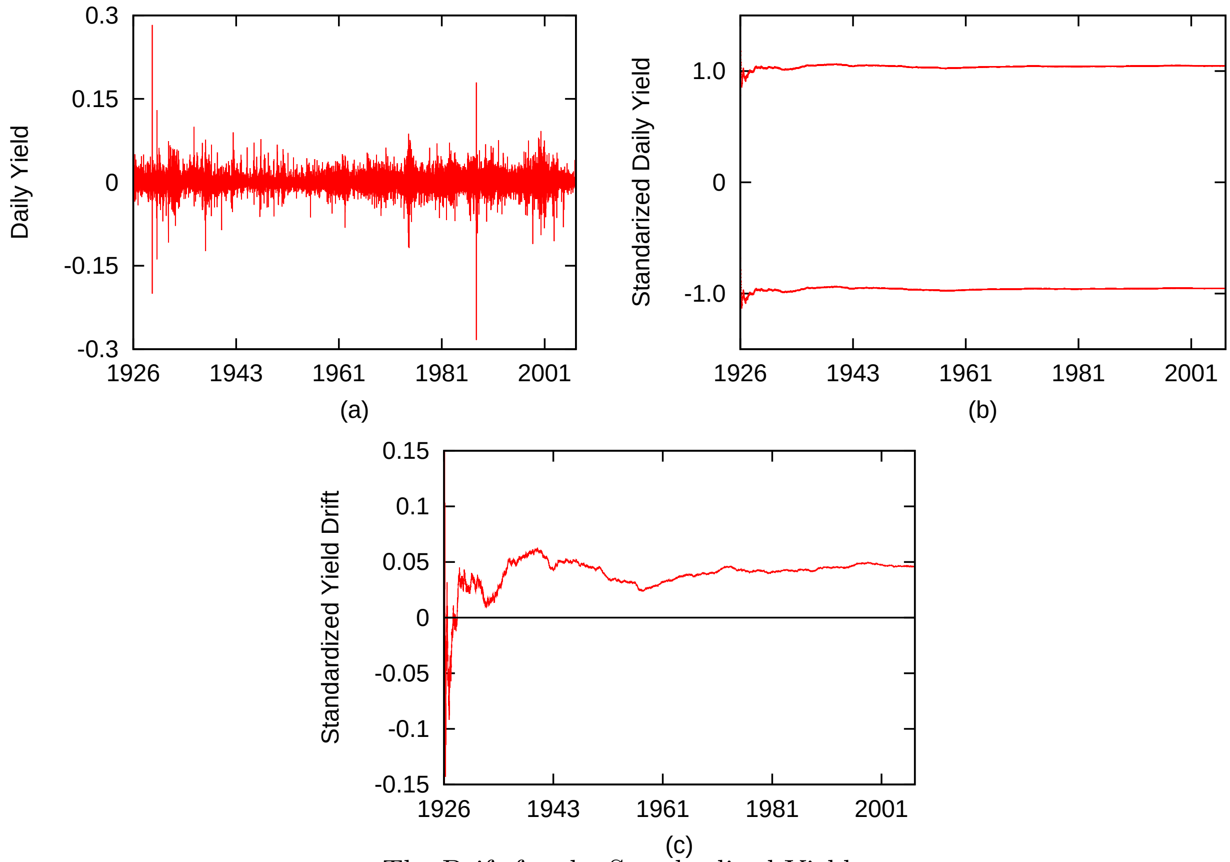
$$\frac{du}{d\tau} = \hat{\mu}(\tau) + \xi(\tau), \tag{7}$$

where

$$\hat{\mu}(\tau(t)) - \mu(t)/\sigma(t) = 0, \tag{8}$$

and ξ is still a Gaussian random variable, but now in τ. Equation (7) is simply a stochastic process with drift $\hat{\mu}(t)$ and unit volatility; its solution in terms of τ is well known. The solution to the original equation, Eq. (3), can then be obtained, at least in principle, by integrating Eq. (6), and then replacing τ with resulting function of t.

In practice, our task is much more difficult. We are *not* given a $\sigma(t)$, and then asked to find the price, $S(t)$, of the stock at subsequent times. We are instead given a collection of stock prices collected over some length of time, and then asked to find the volatility. This is a much more difficult problem, but surprisingly, it is a solvable one, as we will see in the next subsection.



The Drift for the Standardized Yield

FIG. 2: A comparison between the distribution of daily and standardized daily yields for Coca Cola is given in Figs. 2a and b. The binomial behavior of the standardized daily yield can clearly be seen in Fig. 2b. The resultant drift for the standardized daily yield is shown in Fig. 2c.

C. The Discrete Process and a Recursion Relation for $\sigma(t)$

In this subsection, we derive a recursions relation that is used to solve for the volatility as a function of time. This derivation is most conveniently done using a discretized version of the continuous stochastic process Eq. (3) considered above, and we consider $S(t)$ as a continuous approximation to the discrete time series, S_n, for $n = 1, \dots, N_T$, of stock prices collected at equal time intervals, a; this a is usually taken as one trading day. The subscript n enumerates the time step when the price of the stock was collected, and is an integer that runs from 0 to the total number of data points, N_T. As such, $t = na$, $T = N_T a$, $S_n \equiv S(na)$, $u_n \equiv u(na)$, and $\sigma_n \equiv \sigma(na)$ is the volatility at $t = na$. The instantaneous daily yield is

then

$$\frac{du}{dt} \approx \frac{u_n - u_{n-1}}{a} \equiv \frac{\Delta u_n}{a}, \tag{9}$$

where $n \geq 1$. It is clear that $\Delta u_n = \ln(S_n/S_{n-1})$ is the yield of $S(t)$ over the time period a; when a is one trading day, Δu_n is the daily yield.

Our task in this section is to determine σ_n *given* the time-series S_n, and we do so by making use of the analysis in the previous section. We call

$$\frac{\Delta \hat{u}_n}{a} \equiv \frac{\Delta u_n}{a\sigma_n}, \tag{10}$$

the *standardized* yield of the stock price over a time period a, and if a is one trading day, we call it the standardized daily yield. Since

$$\frac{1}{\sigma(t)}\frac{du}{dt} \approx \frac{\Delta u_n}{a\sigma_n}, \tag{11}$$

then from the discretized versions of Eqs. (7) and (8), we see that the distribution of standardized yields has a volatility of $1/a$, or one, if a is set to one trading day. The collection of standardized yields has a *known* volatility.

Consider now a subset of the time-series with $N < N_T$ elements, and the corresponding collection of standardized yields, $\Delta\hat{u}_n$, where n runs now runs from 1 to N. Because this subset was arbitrarily selected from a collection of standardized yields that has a volatility of $1/a$, this subset must also have a volatility of $1/a$. As such

$$\frac{1}{a}\left(1 \pm \sqrt{\frac{2}{N}}\right) = \frac{1}{N-1}\sum_{n=1}^{N}\left(\frac{\Delta u_n}{a\sigma_n} - \frac{1}{N}\sum_{m=1}^{N}\frac{\Delta u_m}{a\sigma_m}\right)^2, \tag{12}$$

where we have included for completeness in Eq. (12) the standard error for the volatility given N data points (see Stuart and Ord 1994) to emphasize that the accuracy of Eq. (12) depends on N. Equation (12) must be true for each $N \leq N_T$. In particular, it must hold for $N-1$, and thus we can write

$$\frac{1}{a}\left(1 \pm \sqrt{\frac{2}{N-1}}\right) = \frac{1}{N-2}\sum_{n=1}^{N-1}\left(\frac{\Delta u_n}{a\sigma_n} - \frac{1}{N}\sum_{m=1}^{N-1}\frac{\Delta u_m}{a\sigma_m}\right)^2, \tag{13}$$

which is similar in form to Eq. (12). This self-similar property of the distribution is used to determine σ_n, as we show below.

We first expand Eq. (12), and single out the $n = N$ terms,

$$\begin{aligned}\frac{1}{a} &= \frac{1}{N-1}\left(\frac{\Delta u_N}{a\sigma_N}\right)^2 + \frac{1}{N-1}\sum_{n=1}^{N-1}\left(\frac{\Delta u_n}{a\sigma_n}\right)^2 \\ &\quad -\frac{N}{N-1}\left(\frac{1}{N}\frac{\Delta u_N}{a\sigma_N} + \frac{N-1}{N^2}\sum_{m=1}^{N-1}\frac{\Delta u_m}{a\sigma_m}\right)^2, \end{aligned} \tag{14}$$

where we have dropped the error terms in Eq. (12) for clarity. Using Eq. (13) in the second term of Eq. (14) and completing a square, we arrive at a surprisingly simple equation for σ_N,

$$\left(\frac{N}{N-1}\right)\frac{1}{a} = \left(\frac{\Delta u_N}{a\sigma_N} - \frac{1}{N}\sum_{m=1}^{N-1}\frac{\Delta u_m}{a\sigma_m}\right)^2. \tag{15}$$

This is easily solved to give,

$$\sigma_N\sqrt{a} = \Delta u_N\left(\frac{1}{N}\sum_{m=1}^{N-1}\frac{\Delta u_m}{a\sigma_m}\sqrt{a} \pm \sqrt{\frac{N}{N-1}}\right)^{-1}, \tag{16}$$

where the sign of the root must be chosen so that $\sigma_n > 0$ for all N. The standardized yield, $\Delta\hat{u}_N$, can then be calculated using Eq. (10) for each time step. Equation (16) gives a recursion relation for σ_N.

A recursive approach to calculating the volatility similar in spirit to the one above is described in Stuart and Ord (1994). That calculation is for volatilities that do not change with time, however, while in ours the volatility can do so explicitly. As we will see below, this introduces a number of complications. We note also that Eq. (16) differs markedly from autoregression approaches such as the EWMA, ARCH, and GARCH in that σ_N depends nonlinearly on Δu_N.

Equation (16) gives a first-order recursion relation for σ_n, and thus given an initial σ_1, the values for σ_n for $n > 1$ is determined. To determine this initial σ_1, we note that in the continuous process Eq. (8) holds. A similar relation must hold for the discretized yields Δu_n.

To determine this relation, we follow the same approach that led to Eq. (12), and consider the following function

$$f_N \equiv \frac{1}{N}\sum_{n=1}^{N}\frac{\Delta u_n}{a\sigma_n} - \frac{1}{\sigma_N}\left(\frac{1}{N}\sum_{N=1}^{N_T}\frac{\Delta u_n}{a}\right). \tag{17}$$

The first term in Eq. (17) is the average of the standardized yield over the first N terms in the time-series, and it corresponds to the discretization of the first term in continuous

constraint Eq. (8). The second term is the quotient of the average daily yield calculated over the same period with the volatility evaluated at the end of this period, and it corresponds to the discretization of the second term in continuous constraint Eq. (8). If σ_1 can be chosen so that the mean of f_N,

$$\frac{1}{N}\sum_{n=1}^{N} f_N, \tag{18}$$

can be minimized to zero at the 95% CL, then Eq. (8) will hold on average for the discretized yield. As usual, the 95% CL for this mean is calculated through the standard error, $[D(f_N)/N_T]^{1/2}$, where

$$D(f_N) \equiv \frac{1}{N_T - 1}\sum_{N=1}^{N_T}\left(f_N - \frac{1}{N_T}\sum_{M=1}^{N_T} f_M\right)^2, \tag{19}$$

is the standard deviation of f_N.

We have successfully applied the recursion relation, Eq. (16), to the 24 DJIA considered here, and have obtained for each stock time-series for σ_n and $\Delta\hat{u}_n$. This was done by determining the quotient $\Delta u_1/\sigma_1$ through an iterative search algorithm that was implemented with a simple C++ program. This algorithm searches for a σ_1 that drives the mean of f_N to zero while in the process minimizing $D(f_N)$. In addition, since the volatility must be non-negative, this search is done under the constraint that all calculated values for σ_n must be greater or equal to zero, and it was stopped once the mean of f_N has been calculated to sufficient accuracy.

A $\Delta u_1/\sigma_1$ that minimizes f_N while at the same time giving a non-negative value for the volatility can be found for all 24 stocks. Indeed, we found that the mean of f_N can be driven as close to zero as needed. The results of this calculation is given in Table II, which lists for each stock the value of $\Delta u_1/\sigma_1$, the mean of f_N for this $\Delta u_1/\sigma_1$, and the standard error of the mean. While values for $\Delta u_1/\sigma_1$ is only given to an accuracy of 10^{-7}—which is sufficient given the accuracy of the S_n for the stocks as noted in Appendix A—we have been able to drive the mean value of f_N to as far down as 10^{-16} by increasing the accuracy of $\Delta u_1/\sigma_1$ to 10^{-15}. It is clear from the standard errors given in Table II that the mean of f_N vanishes within standard error at the 95% CL. This validates the recursion relation for σ_n for all 24 stocks.

Implicit in the derivation of Eq. (12) is that N is large, and yet since σ_N starts at some initial point σ_1, σ_N are necessarily generated at small N. We would thus expect that there

is a transient interval marked by some $N_{tran} < N_T$ for which the solution to Eq. (16) for $N < N_{tran}$ is markedly different from the solution when $N > N_{tran}$. This is seen. For all 24 stocks, the behavior of $\Delta\hat{u}_n$ for n near one is different than its behavior for large n. This difference is similar for all of the stocks, indicating that it is due to the recursion process itself, and not to any underlying behavior of the markets. We would thus hesitate to use the calculated values for σ_n when $n < N_{tran}$ to draw conclusions about the behavior of the stock. The length of this interval, N_{tran}, varies from stock to stock, but typically ranges between 100 to 400 trading days. Given that the shortest time-series considered here contains 5,088 trading days, this interval is extremely short for all 24 stocks, and is not relevant in practice. This is yet another reason why we have chosen stocks that have a long track record to analyze.

TABLE II: Determining σ_1 for the 24 DJIA Stocks

	$\Delta u_1/\sigma_1$	Mean f_N	SE for f_N
GE	-1.29986151	3.70×10^{-12}	2.51×10^{-6}
AXP	-1.11059569	-4.56×10^{-7}	7.60×10^{-6}
PFE	-0.69103252	6.37×10^{-9}	6.02×10^{-6}
DIS	-0.62361094	-1.66×10^{-11}	1.08×10^{-5}
MSFT	-0.23500530	-4.61×10^{-11}	3.36×10^{-5}
KO	-0.00633264	8.28×10^{-12}	3.39×10^{-6}
PG	0.29288976	-2.68×10^{-5}	2.47×10^{-2}
GM	0.29925834	1.19×10^{-11}	5.08×10^{-6}
AIG	0.30999620	-2.73×10^{-11}	5.90×10^{-6}
MMM	0.36915581	-4.67×10^{-11}	5.32×10^{-6}
AA	0.37469224	1.30×10^{-12}	4.77×10^{-6}
HPQ	0.64209911	1.55×10^{-11}	6.24×10^{-6}
JNJ	0.96565296	-1.96×10^{-11}	5.54×10^{-6}
CAT	0.98788045	-2.20×10^{-11}	5.27×10^{-6}

continued from previous page

	$\Delta u_1/\sigma_1$	Mean f_N	SE for f_N
INTC	0.99040135	-6.99×10^{-12}	1.58×10^{-5}
WMT	0.99093623	5.31×10^{-8}	1.53×10^{-5}
C	1.05691370	1.41×10^{-10}	5.76×10^{-6}
IBM	1.12634844	-1.08×10^{-11}	5.73×10^{-6}
MO	1.18101555	1.15×10^{-11}	1.35×10^{-6}
VZ	1.30751627	2.03×10^{-7}	1.40×10^{-5}
DD	1.34345528	-4.88×10^{-12}	1.17×10^{-5}
BA	1.41323601	-2.74×10^{-11}	1.36×10^{-5}
XOM	1.41414287	-1.88×10^{-7}	5.08×10^{-3}
MRK	1.41417225	-9.05×10^{-7}	7.72×10^{-3}

V. THE STANDARDIZED DRIFT AND ITS DISTRIBUTION

Although the recursion relation, Eq. (16), has been successfully solved for all 24 stocks, we will delay until Sec. VI to present the solutions to this equation. Instead, we will first validate our model by showing that the stochastic process introduced in the previous sections solves the overly large kurtosis problem raised in Sec. IV. In the process, we will find that the distribution of standardized yield is a generalized Rademacher distribution, and will show that the simple skewness and kurtosis agree with the values for population skewness and kurtosis for this distribution. By doing so, we will also have validated our model of stock market prices using the criteria outlined in Sec. III. As part of this process, we will be able to determine the drift of the yield as a function of time as well.

A. Observed Properties of the Standardized Yield

To complement the autoregression calculation for the daily yield shown in Fig. 1, we have calculated autocorrelation function for the standardized daily yield, $G^{(2)}(\Delta\hat{u}_{N_T}, \Delta\hat{u}_{N_T-M})$, for all 24 DJIA stocks. A plot of $G^{(2)}(\Delta\hat{u}_{N_T}, \Delta\hat{u}_{N_T-M})$ as a function of $T = aM$ has the same shape as that shown in Fig. 1, but with $G^{(2)}(\Delta\hat{u}_{N_T}, \Delta\hat{u}_{N_T}) = 1$ for all the stocks

instead of a range of values. Like the daily yield, the standardized daily yield on any one day is not correlated with the standardized yield on any other day; this is to be expected if the volatility is a function of time only. We have also determined the number of trading days for which the value of the autocorrelation function falls within the 95% and 99% CI of zero. The results are shown in Table I, and for for all but one stock, the results as as expected. On 5% of the trading days, the value of the autocorrelation function exceeds the 95% CI, and on 1% of the trading days, the value exceeds the 99% CI. The only exception is Microsoft at the 95% CL when on 6% of the trading days the autocorrelation function exceeds the 95% CI of zero.

In Sec. IV, we noted that on at least 1% of the trading days the value of the autocorrelation function for the daily yield exceeds either the 95% or 99% CI of zero, and we can say with a degree of statistical certainty that for these days, the autocorrelation function does not vanish. With the exception of Microsoft at the 95% CL, such days are not found in the autocorrelation function of the standardized yields. Since the standardized yield is obtained from the yield by removing the time-dependent volatility, we conclude from the results in Table I for the standardized yield that, with the possible exception of Microsoft, this 1% is not due to correlations in the daily yield, but rather to temporal variations in the volatility.

Next, shown in Fig. 2a is a plot of the daily yield with respect to trading day for Coca Cola. In comparison, Fig. 2b is the is the plot of the standardized daily yield for the stock over the same period. It is readily apparent that instead of taking a range of values between ± 0.3 as the daily yield does, the standardized yield jumps between two values, one near $+1$ and one near -1. Notice also that while the standardized yield is not precisely $+1$ or -1, any changes in the standardized yield near $+1$ are accompanied by the *same* variations of the yield near -1; the variations in the standardized yield near $+1$ and near -1 would seem to move up or down in unison. Indeed, using a 251-day moving average, we find that the average of the difference in the value of the standardized daily yield, $A_n^{(+)}$, near $+1$ and its value, $A_n^{(-)}$, near -1 ranges from a minimum of $(A^{(+)} - A^{(-)})/2 = 0.9934 \pm 0.0061$ to a maximum of $(A^{(+)} - A^{(-)})/2 = 1.0083 \pm 0.0068$; both are within the 95% CI of one.

This binomial behavior for the yield is not surprising for the same reasons that binomial trees are effective at pricing options. As noted by Cox and Ross (1976), a continuous stochastic process with constant volatility can be approximated as a discrete random walk where at each time step, na, there is a probability, p, that the stock price will increase at the

next time step, and a probability $1-p$ that it will decrease. The discrete stochastic process can thus be approximated by a binomial distribution, and as the binomial distribution is known to approach the Gaussian distribution in the large n limit, the discrete random walk approaches a continuous stochastic process for the stock price. Indeed, this limit is the reason why binomial trees are effective in the first place.

B. Determining the Drift for the Standardized Yield

In this subsection, we will determine the drift of the standardized daily yield as a function of time. We do so by noting that it is apparent from Fig. 2 that the distribution of standardized daily yields is a binomial distribution. Thus, at each time step, n, there is a probability, p, that the standardized yield will increase by an amount $A_n^{(+)}$ on that day, and probability, $1-p$, that it will decrease by an amount $A_n^{(-)}$. While in principle p may be different at different time steps, the fact that $A_n^{(+)}$ and $A_n^{(-)}$ change in unison while keeping the average distance between position and negative standardized yields constant suggests that any variation in time is due to an overall shift in the distribution. Variations in $A_n^{(+)}$ and $A_n^{(-)}$ are not due to a time-dependent p, but rather to a drift for the standardized yield that changes with time.

With this realization, the drift can easily be determined for all 24 stocks. From Eqs.(7) and (11), we can express the standardized daily yield as

$$\Delta\hat{u}_n = \hat{\mu}_n + \xi_n^R, \tag{20}$$

where $\hat{\mu}_n \equiv \hat{\mu}(an)$ is the discretized drift of the standardized yield, and ξ_n^R is a random variable with zero mean and unit volatility such that $E_R[\xi_n^R \xi_m^R] = \delta_{nm}$. While for the continuous process ξ_R would be a Gaussian random variable, for the discrete process we will show that ξ_n^R is a random variable for the generalized Rademacher distribution described below.

The standard way of calculating $\hat{\mu}_n$ is to use a moving average over a window of M days. However, just like for the volatility, calculating $\hat{\mu}_n$ with a moving average will mean that variations in the drift faster than M cannot be clearly seen. We will instead calculate $\hat{\mu}_n$ directly from $\Delta\hat{u}_n$, which is possible to do because the distribution of standardized yield is so simple.

We first note that since changes to $A_n^{(+)}$ and $A_n^{(-)}$ is due to shifts in the distribution of standardized yield with time, these shifts must be due to the drift, $\hat{\mu}_n$, of the standardized yield. Shifts in random variables are trivial changes to the distribution, however, and a drift that changes with time will not materially change the distribution of standardized yields.

We next note that $E_R[\xi_n^R] = 0$. As the values of $\Delta \hat{u}_n$ lie close to ± 1, we conclude that ξ_n^R can only take the values ± 1. Any deviation by $A_n^{(\pm)}$ from ± 1 must be due to the drift. This drift can be determined by solving the equation

$$\hat{\mu}_n = \Delta \hat{u}_n - \xi_n^R, \tag{21}$$

for $\hat{\mu}_n$ by taking the sign of ξ_n^R to be the same as the sign of $\Delta \hat{u}_n$. This solution is straightforwardly implemented, and the results for Coca Cola is shown in Fig. 2c. For clarity, we have only shown the values of the drift between ± 0.15. While there are values that lie outside of this range, they occur in the first 10 time steps in the series, and are part of the transient behavior mentioned above.

The drift for the standardized yield of all 24 stocks have be found using this approach. Not surprisingly, we find that 21 out of the 24 stocks have a drift that positive for the great majority of the time-series. What is surprising is that for three of the 24 stocks (Exxon-Mobil, Merck, and Proctor and Gamble) the drift of the yield of the stock is *negative* outside the transient region. For these stocks, the only reason why their price increases is due to the random walk, and because the volatility is so much greater than the drift.

C. The Distribution of Standardized Daily Yields

While in the last subsection we determined the drift of the daily yield, in this subsection we will show that the distribution of the standardized daily yield is a generalized Rademacher distribution shifted by the drift, $\hat{\mu}_n$. This will be done by comparing the skewness and kurtosis of the Rademacher distribution with the sample skewness and kurtosis for $\Delta \hat{u}_n$ after the drift has been removed. We will show that the two agree at the 95% CL, and doing so will both determine the distribution and validate our model of stock market prices as a stochastic process with a time-dependent volatility. We begin by describing the properties of the generalized Rademacher distribution.

A generalize Rademacher distribution consists of a random variable, ξ^R, that takes the

value $+1$ with probability p, and the value -1 with probability $1-p$. We denote the expectation value for this distribution as $E_R[\cdot]$, and find that the population mean, $mom'_1 = E_R[X]$, is simply

$$mom'_1 = p - (1-p) = (2p-1). \tag{22}$$

This vanishes for $p = 1/2$. The kth population moment, $mom_k \equiv E_R[(\xi^R - E_R[\xi^R])^k]$, is easily calculated to be

$$mom_k = (2)^k p(1-p)\left[(1-p)^{k-1} + (-1)^k p^{k-1}\right]. \tag{23}$$

The population variance is thus

$$mom_2 = 4p(1-p) \tag{24}$$

, while the population skewness is

$$Skew = \frac{1-2p}{\sqrt{p(1-p)}}, \tag{25}$$

and the population kurtosis for the distribution is

$$Kurt = \frac{1-3+3p^2}{p(1-p)}. \tag{26}$$

Clearly, if $p = 1/2$, then $m_2 = 1$, $Skew = 0$, and $Kurt = 1$; this is the Rademacher distribution, which is a special case of the binomial distribution. When $p \neq 1/2$, we call this the generalized Rademacher distribution.

Given the plot in Fig. 2b, we would expect that the distribution of standardized yields to be a Rademacher distribution with $p = 1/2$ at all time steps. To show that that this is the case, we have calculated the sample skewness and the kurtosis of the standardized yield after the drift, $\hat{\mu}_n$, has been removed from $\Delta\hat{u}_n$. This has been done for all 24 stocks using the entire time-series for each. We then compared these sample skewness and kurtosis with the population skewness and kurtosis for the Rademacher distribution using the t-Test. For completeness, we have also calculated the probability, p, for each stock by counting the total number of $\Delta\hat{u}_n > 0$, and compared it to the Rademacher value of $p = 1/2$ using the chi-squared test. The results of these calculations and tests are given in Table III. We see that for all but four of the stocks the fit is exceedingly good; the skewness, the kurtosis, and the probability all agree at the 95% CL.

TABLE III: Model Validation

	Skewness		Kurtosis		Probability	
	Mean	t-Test	Mean	t-Test	p	χ^2
GM	0.001 ± 0.014	0.07	1.00028 ± 0.00030	0.92	0.500	0.01
DD	0.002 ± 0.014	0.15	1.00028 ± 0.00031	0.92	0.499	0.02
VZ	-0.005 ± 0.026	0.18	1.0011 ± 0.0012	0.92	0.501	0.03
DIS	0.010 ± 0.018	0.54	1.00058 ± 0.00063	0.92	0.498	0.29
AXP	0.013 ± 0.022	0.58	1.00086 ± 0.00093	0.92	0.497	0.34
MMM	-0.011 ± 0.016	0.66	1.00050 ± 0.00054	0.93	0.503	0.43
PG	-0.009 ± 0.014	0.67	1.00038 ± 0.00041	0.93	0.502	0.45
JNJ	-0.012 ± 0.016	0.78	1.00053 ± 0.00056	0.94	0.503	0.60
HPQ	-0.015 ± 0.019	0.78	1.00072 ± 0.00078	0.94	0.504	0.61
KO	-0.018 ± 0.013	0.94	1.00044 ± 0.00046	0.96	0.503	0.89
C	-0.028 ± 0.028	0.98	1.0019 ± 0.0020	0.97	0.507	0.96
BA	0.019 ± 0.015	1.28	1.00066 ± 0.00064	1.04	0.495	1.64
AIG	-0.028 ± 0.022	1.32	1.0015 ± 0.0014	1.04	0.507	1.73
INTC	-0.029 ± 0.022	1.34	1.0015 ± 0.0015	1.05	0.507	1.79
PFE	-0.022 ± 0.016	1.42	1.00087 ± 0.00081	1.07	0.506	2.01
CAT	-0.020 ± 0.014	1.42	1.00069 ± 0.00064	1.07	0.505	2.01
GE	-0.020 ± 0.014	1.49	1.00069 ± 0.00063	1.10	0.505	2.21
AA	0.025 ± 0.170	1.50	1.00107 ± 0.00098	1.70	0.494	2.26
MSFT	-0.049 ± 0.028	1.76	1.0035 ± 0.0030	1.19	0.512	3.12
WMT	-0.038 ± 0.022	1.77	1.0022 ± 0.0018	1.19	0.510	3.13
MRK	-0.037 ± 0.016	2.32	1.0018 ± 0.0013	1.40	0.509	5.37
IBM	-0.037 ± 0.014	2.68	1.0016 ± 0.0010	1.55	0.509	7.21

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	Skewness		Kurtosis		Probability	
	Mean	t-Test	Mean	t-Test	p	χ^2
XOM	-0.039 ± 0.014	2.86	1.0018 ± 0.0011	1.63	0.510	8.20
MO	-0.040 ± 0.014	2.94	1.0019 ± 0.0011	1.66	0.510	8.67

Although this agreement is not as good for Altria, Exxon, IBM, and Merck, this is only because we were comparing it with the Rademacher distribution with $p = 1/2$. We find that the distribution of standardized yields for these stocks is instead the generalized Rademacher distribution with a p slightly greater than 1/2. Using Eq. (22) and the sample mean for these stocks, we have solved for a probability, p, for the stocks. We find that this probability is in close agreement with those listed in Table III for these stocks, and when this p is then used in Eqs. (25) and (26) to predict values for the skewness and kurtosis, the predicted values are now in agreement with the sample skewness and kurtosis at the 95% CL.

We also note that variance of the distribution calculated using the values for p given in Table 4 in Eq. (24) ranges from 0.9994 to 1.000. This is in excellent agreement with the requirement that the variance of the standardized daily yield is one when a is one trading day.

We thus conclude that the distribution of standardized daily yields is a generalized Rademacher distribution shifted by the drift, $\hat{\mu}_n$. For 20 of these stocks, we find that $p = 1/2$ at the 95% CL. The probability that the daily yield increases is the same as the probability that it decreases. For the other four stocks, p is slightly greater than 1/2, and the probability that the daily yield increases is slightly larger than the probability that it decreases.

VI. THE INSTANTANEOUS VOLATILITY

Having determined the distribution for the standardized daily yield, we now turn our attention to determining the volatility of the stock.

We find that while the recursion relation, Eq. (16), is straightforwardly solved using the σ_1 given in Table II, there is a great deal of noise associated with the resultant values for σ_n. This can be seen in Fig. 3a where we have plotted as a function of trading day the volatility

obtained from Eq. (16). Although we can discern that there is an inherent structure in graph, this structure is buried within random fluctuations of σ_n. These fluctuations are due to random noise generated when Eq. (16) is solved, and they mask the functional dependence of σ on t. In this section, we will extract this dependence from the noise.

The presence of the noise in σ_n is inherent, but not because $\sigma(t)$ itself obeys a stochastic process, as is assumed in stochastic volatility models. If it were, then there will necessarily be a second stochastic differential equation for $\sigma(t)$ to augment Eq. (7), and the two coupled equations would have to be solved simultaneously. Certainly, Eq. (7) and the recursion relation Eq. (16) would not, in general, be solutions of the coupled stochastic differential equations, and it is this recursion relation that was used to obtain Fig. 3a. Rather, this noise is inherent in determining the volatility itself.

Note from Eq. (16) that $\sigma_n \propto \Delta u_n/a$. For a stochastic process of the form Eq. (3) where the volatility changes with time, at each time step, an, Δu_n is a random variable from a distribution with volatility σ_n. As σ_n need not equal σ_m for any two n and m, each Δu_n can come from a *different* distribution. In the worst case, we will have only *one* Δu_n out of any distribution with which to determine σ_n, and this Δu_n can take any value from $-\infty$ to $+\infty$ with a probability

$$P(\Delta u_n/a) = \frac{1}{\sigma_n}\sqrt{\frac{a}{2\pi}}e^{-(\Delta u_n/a-\mu_N)^2 a/2\sigma_n^2}. \tag{27}$$

Determining σ_n would thus seem to be an impossible task. That it can nevertheless be done is due to three observations. First, because $P(\Delta u_n/a)$ is Gaussian, there is a 68% probability that any value of $\Delta u_n/a$ will be within $\mu_n \pm \sigma_n/\sqrt{a}$. It is for this reason that it is still possible to discern an overall functional dependence of σ_n on the trading day through the noise in Fig. 3. Second, $\sigma(t)$ is a *deterministic* function of t, and thus the value of the volatility at time step an is related to its value at time step $a(n-1)$. Given a sufficient number of Δu_n—and thus a sufficient number of σ_n—it must be possible to construct a functional form for $\sigma(t)$. Third, using Fourier analysis (also called spectral analysis) and signal processing techniques, it is possible to remove from Fig. 3 the noise that is obscuring the details of how σ_n depends on the trading day, and obtain a functional form for the volatility.

That Fourier analysis provides an efficient way of removing the noise from Fig. 3 is based on the following theorem:

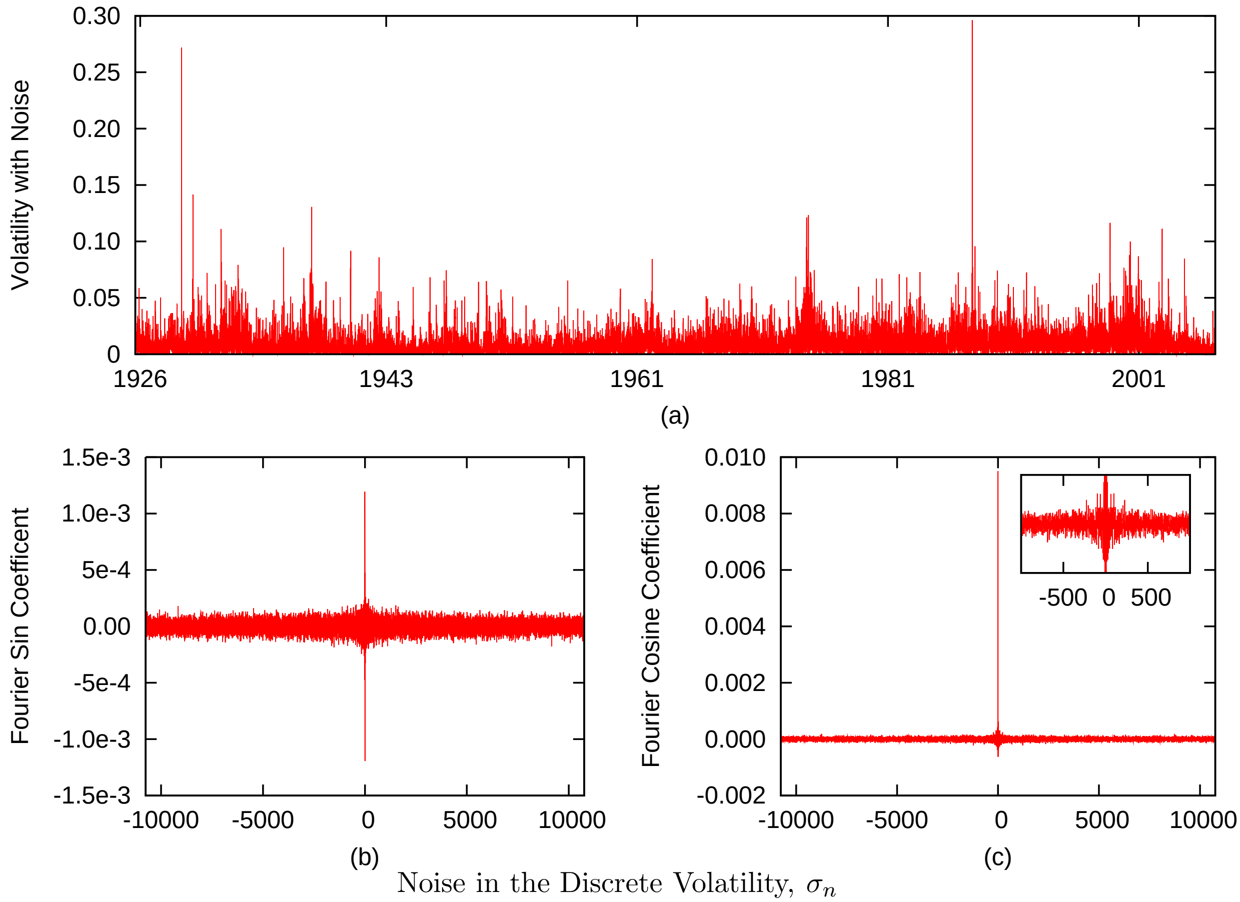


FIG. 3: The top figure shows the volatility for Coca Cola obtained from the recursion relation Eq. (16). The high degree of noise associated with this volatility can readily be seen. In Figs. 3b and c, the Fourier sine and cosine components are graphed, and the floor of noise for both can readily be seen along with the points that are above the noise. From the insert in Fig. 3c, the similarity in the noise floors for the sine and cosine coefficients is apparent.

Theorem: If $\{\xi_n : n = 1, \ldots, N\}$ is a time series where ξ_n is a Gaussian random variable with zero mean, and volatility, σ, then the Fourier sine, $\alpha_k^{\sin}$, and Fourier cosine, $\alpha_k^{\cos}$, coefficients of the Fourier transform of ξ_n are Gaussian random variables with zero mean and volatility $\sigma/\sqrt{N}$.

This theorem is well-known in signal analysis, and is an immediate consequence of Parseval's Theorem. A proof of this theorem, as well as a review of the discrete Fourier transform, is given in Appendix C. It is because the volatility of the Fourier sine and cosine coefficients

for Gaussian random variables are reduced by a factor of $1/\sqrt{N}$ that it is possible to remove from σ_n the random noise. In general, this reduction in the coefficients does not occur if the ξ_n are *not* random variables, and thus the structure in Fig. 3 can be resolved once the Fourier transform of σ_n is taken. After this removal is accomplished, we can then take the inverse Fourier transform to obtain $\sigma(t)$, which we call the instantaneous volatility to differentiate it from the σ_n that comes directly from Eq. (16).

Figure 3b and c are plots of the Fourier sine and Fourier cosine coefficients of the discrete Fourier transform of σ_n defined as

$$a_k^{\cos} = \frac{1}{N_T}\sum_n^{N_T} \sigma_n \cos\left(\frac{2\pi ikn}{N_T}\right), \qquad a_k^{\cos} = \frac{1}{N_T}\sum_n^{N_T} \sigma_n \sin\left(\frac{2\pi ikn}{N_T}\right). \tag{28}$$

They depend on an integer k, which runs from $-(N_T-1)/2$ to $(N_T-1)/2$. As

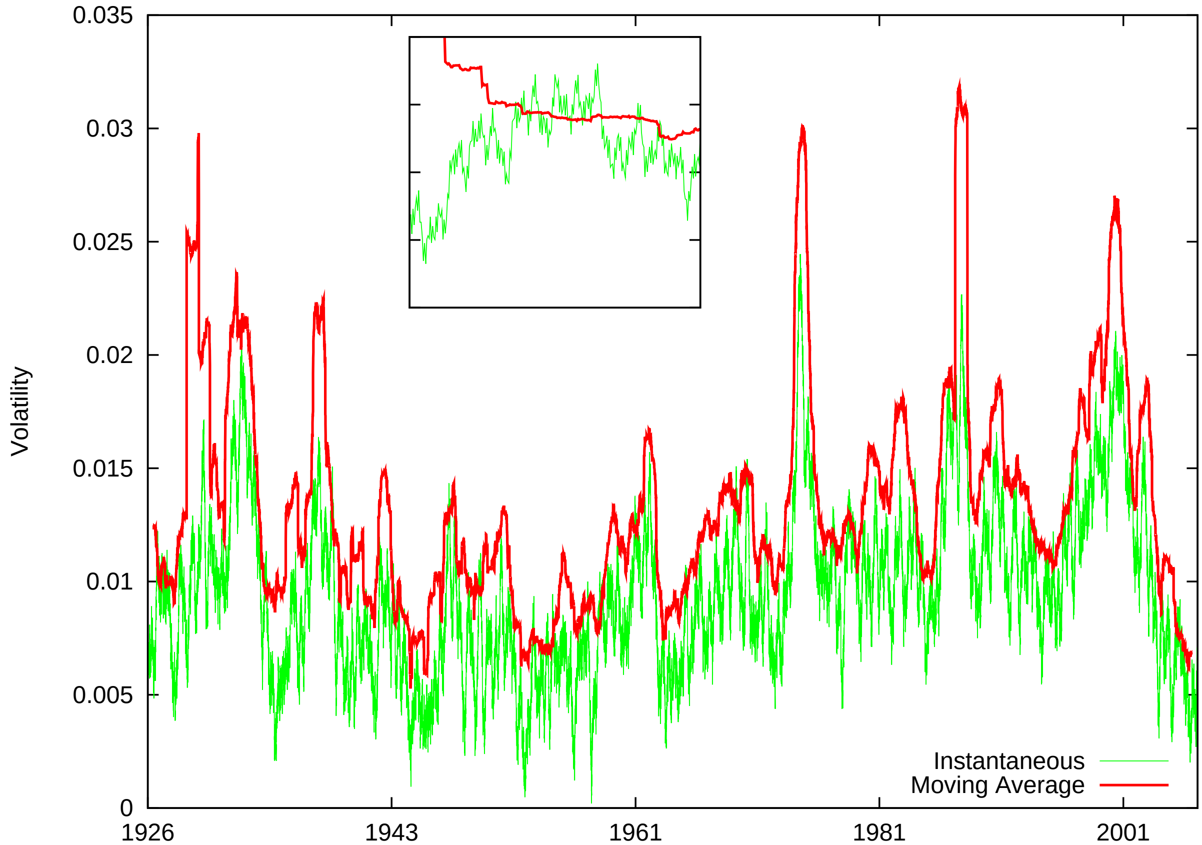
$$\sigma_n = a_0^{\omega} + 2\sum_{k=1}^{(N_T-1)/2} a_k^{\cos}\cos\left(\frac{2\pi kn}{N_T}\right) + 2\sum_{k=1}^{(N_T-1)/2} a_k^{\sin}\sin\left(\frac{2\pi kn}{N_T}\right), \tag{29}$$

the Fourier transform decomposes the time-series, σ_n, into components that oscillate with frequency $f_k = k/N_T$ day^{-1} (or, equivalently, with period N_T/k days) for $k > 0$; the coefficients $2|a_k^{\cos}|$ and $2|a_k^{\sin}|$ are the amplitudes of these oscillations.

In the graphs shown in Figs. 3b and3c, we can readily see that there is a component of the Fourier coefficients for Coca Cola that varies randomly between ± 0.0002. This is the noise floor. Coefficients in this floor are the result of the Fourier transform of the noise that mask the functional behavior of σ_n on the trading. This noise floor is similar for both the Fourier sine and cosine coefficients, as can be see in detail in inset plot in Fig. 3c where the features of the plot of $a_k^{\cos}$ are magnified for k between ± 1000.

It is also apparent from the graph that there are Fourier coefficients that rise above the noise. While the most prominent of these is $a_0^{\cos}$ (which is the average of x_n over all trading days), such points exist for other coefficients as well. This is due to the structure in σ_n shown in Fig. 3a; if there were no structure at all in the plot, then there would be no Fourier coefficients that rise above the noise floor.

By combining this observation with the near uniformity of the noise floor, we are able to filter out the noise component of σ_n, and construct a (approximately) noise-free instantaneous volatility, $\sigma(t)$. A description of the process that we used, along with the statistical criteria used to determine the noise floor for the Fourier sine and cosine coefficients, is given



The Instantaneous Volatility, $\sigma(t)$, After Noise Removal

FIG. 4: In the main figure, the instantaneous volatility Coca Cola and the historical volatility the stock calculated with a 251-day moving average is plotted. The historical volatility consistently overestimates the instantaneous volatility. The degree of this overestimation, along with the details that the moving average misses, can be readily seen in the inset graph where both volatilities are plotted over a one-year span from December 2, 2004 to December 01, 2005.

in detail in Appendix C 3. The effectiveness of the noise removal process can be seen in Fig. 4 where a plot of the instantaneous volatility for Coca Cola is shown. When this plot is compared to Fig. 3a, the amount of noise removed, and the success of the noise removal procedure, is readily apparent. Indeed, out of a total of 21,523 Fourier sine and cosine coefficients for σ_n, 10,734 Fourier sine and 10,728 of Fourier cosine coefficients were removed as noise; only 59 points were kept to construct $\sigma(t)$. While the graph of $\sigma(t)$ may appear to be noisy, this is because eight decades of trading days are plotted in the figure. Much of this apparent noise disappears when the range of trading days plotted is narrowed, as can

be seen in the inset figure. Here, the instantaneous volatility over a one-year period from December 29, 2005 to December 29, 2006 has been plotted.

To compare the instantaneous volatility with the historical volatility, we have included in Fig. 4 a graph of historical volatility calculated from the daily yield using a 251-day moving average. It is immediately apparent that the historical volatility is generally larger than the instantaneous volatility; at times it is dramatically so. It is also readily apparent that the historical volatility does not show nearly as much detail as the instantaneous volatility, as can be seen in the inset figure.

A functional form for $\sigma(t)$ can be found for all 24 stocks. For Coca Cola, this expression has 59 terms; we give only four of them here,

$$\begin{aligned}\sigma(t) \; = \; 0.00950 \Bigg[& 1 - 0.25126 \sin(2\pi f_0 t) + 0.12670 \cos(2\pi f_0 t) \\ & +, \ldots, + 55 \text{ terms} +, \ldots, + 0.03870 \cos(8735[2\pi f_0]t) \Bigg],\end{aligned} \tag{30}$$

where $f_0 = 1/21522$ rad/day is the fundamental angular frequency. The amplitude of the first term in the expression is the largest; it is the average of σ_n over all the trading days in the time-series. The second largest amplitude is the sine term in Eq. (30), and it is 25% the size of the first. All other amplitudes are smaller then this term, for most by a factor of 5, and yet notice from Fig. 4 that these amplitudes are nonetheless sufficient to generate a instantaneous volatility that is far from a constant function.

From the last term in Eq. (30), we see the that shortest frequency of oscillations that make up $\sigma(t)$ is $8735/21522 \approx 0.4$ day^{-1}. This is very close to the Nyquist criteria of 0.5 day^{-1} for $\sigma(t)$, which is the upper limit on the frequencies of the Fourier components of $\sigma(t)$. The underlying reason for such a limit is because the the original time-series, S_n, was acquired once each trading day. We therefore cannot *measure* oscillations with a period shorter than two trading days; there simply is not enough information about the stocks to determine what happens within the trading day. (This is in contrast to *predicting* how the volatility may behave during the trading day, which certainly can be done.) For each of the 24 stocks, the shortest period of the Fourier components that make up the instantaneous volatility are listed in Table III, and we see that for all but 3 of the stocks our expression for $\sigma(t)$ comes very close to Nyquist criteria. In the case of Alcoa, Caterpillar, and Johnson & Johnson, the shortest period has even reached it.

TABLE IV: Effectiveness of Noise Filtering Routine

	Period (days)	$a_k^{\sin}$ Noise Floor $\times$ SD	$a_k^{\sin}$ Noise Kurtosis	$a_k^{\cos}$ Noise Floor $\times$ SD	$a_k^{\cos}$ Noise Skewness	$a_k^{\cos}$ Noise Kurtosis
CAT	2.0	3.33	3.07	3.36	-0.04	3.04
JNJ	2.0	3.20	2.99	3.23	-0.03	2.99
AA	2.0	3.40	3.00	3.47	-0.02	3.00
GE	2.1	3.47	3.01	3.44	-0.04	3.00
HPQ	2.1	3.45	3.00	3.31	-0.08	3.01
DIS	2.1	3.42	2.99	3.64	-0.03	3.03
XOM	2.2	3.79	3.11	3.61	-0.05	3.03
MSFT	2.3	3.51	2.99	3.27	-0.08	3.00
PFE	2.3	3.38	3.00	3.40	-0.05	3.00
AXP	2.4	3.56	3.17	3.33	-0.04	3.14
MRK	2.4	3.33	3.03	3.95	-0.03	3.00
INTC	2.4	3.26	3.00	3.36	-0.05	2.99
WMT	2.5	3.47	2.96	3.29	0.11	2.99
KO	2.5	3.89	3.30	3.53	0.02	3.23
BA	2.7	3.37	3.00	3.30	-0.01	2.99
PG	3.2	3.31	2.95	3.40	-0.04	3.00
AIG	3.2	3.44	3.00	3.32	0.01	3.00
MMM	3.5	3.03	2.54	3.06	-0.04	2.58
IBM	4.0	3.41	3.00	3.64	0.03	3.00
VZ	5.2	3.28	2.99	3.23	-0.09	3.00
MO	5.5	3.56	3.14	3.64	-0.07	3.21

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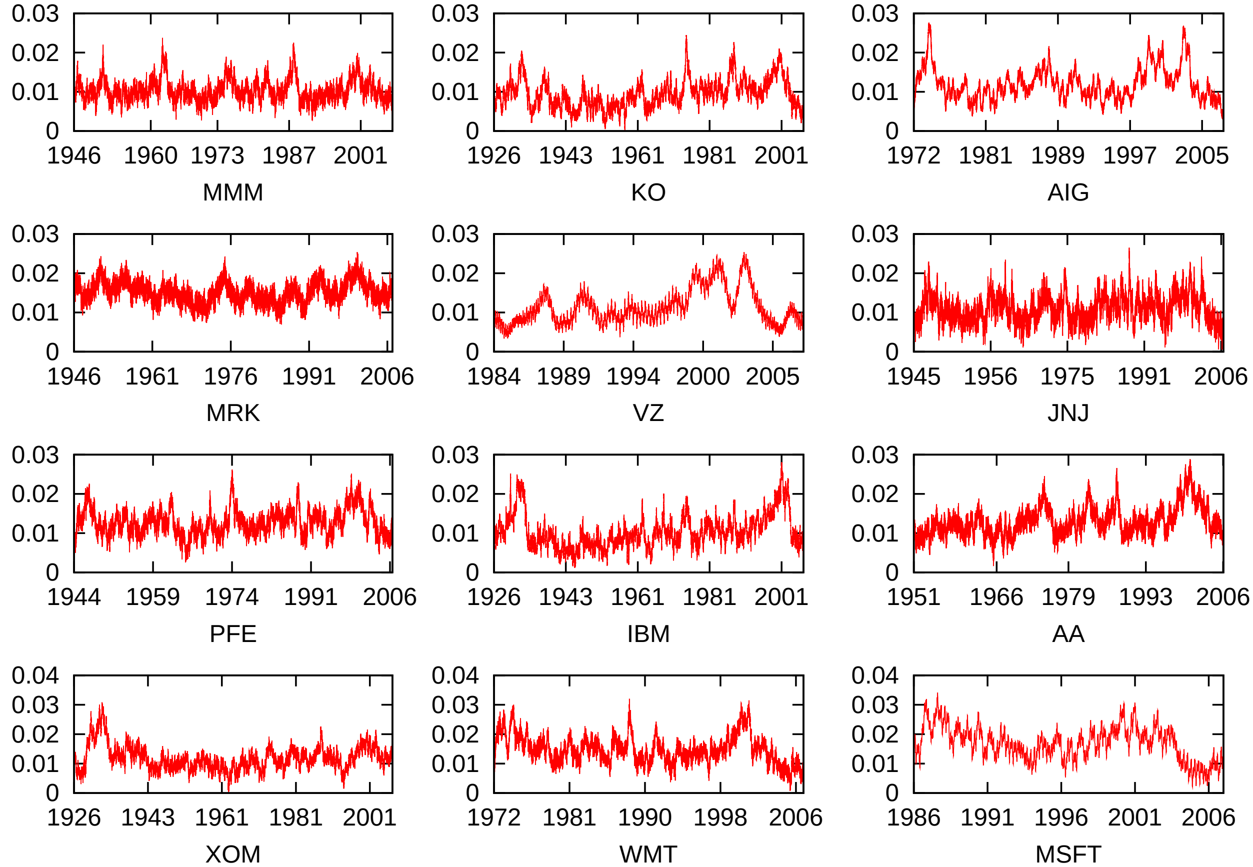
	Period (days)	$a_k^{\sin}$ Noise Floor $\times$ SD	$a_k^{\sin}$ Noise Kurtosis	$a_k^{\cos}$ Noise Floor $\times$ SD	$a_k^{\cos}$ Noise Skewness	$a_k^{\cos}$ Noise Kurtosis
C	6.2	3.36	3.02	3.33	0.05	3.12
DD	9.6	3.81	3.31	3.64	0.00	3.12
GM	14.2	3.59	3.00	3.64	-0.06	3.00

In Figs. 5 and 6, we have graphed the instantaneous volatility as a function of trading day for all 24 stocks. They have been ordered into graphs where the degree of volatility are similar, with the stocks with roughly the highest volatility graphed last. Analytical expressions for the other 23 stocks are not given as they are too lengthy.

With $\sigma(t)$ now known and the drift for the standardized daily yield obtained previously, the drift for the daily yield, $\mu_n \equiv \mu(na)$, can be found for all 24 stocks using the discretized version of Eq. (8), $\mu_n = \sigma(na)\hat{\mu}_n$, where the $\sigma(t)$ is the instantaneous volatility obtained above. Since $|\hat{\mu}_n| < 1$ for all 24 stocks, $|\mu_n| < \sigma(na)$. Thus for all of the 24 stocks, the drift of the stock is smaller than the volatility of it. This is to be expected. If the drift of a stock is larger than the volatility, then future trends in the stock can be predicted with a certain degree of certainty; the drift is, after all, a *deterministic* function of time. Such trends could be seen by investors, and nearly riskless profits could be made. This clearly does not happen. It is instead very difficult to discern future trends in the price of stocks, and this is precisely because the volatility of the stock is so large.

VII. CONCLUDING REMARKS

As a continuous process, we have found that the 24 DJIA stocks can be described as a stochastic process with a volatility that changes deterministically with time. It is a process for which the autocorrelation function of the yield vanishes at different times, and thus one that describes a stock whose price is efficiently priced. From the results of our calculation of the autocorrelation function of the daily yield for the 24 stocks, this property of our stochastic process is in very good agreement with how these stocks are priced by the market. It is also a process for which the solution of the stochastic differential can be, at least formally,



The Instantaneous Volatility for the DJIA Stocks, I

FIG. 5: Graphs of the instantaneous volatilities verses trading day for the 12 of the 24 DJIA stocks with the lowest peak volatility are shown.

solved. This solution is valid only because the volatility is a deterministic function of time, however. If the volatility is depends on the stock price, or if the volatility itself is a stochastic process, the solution of the stochastic differential equation will not be so simple, and the autocorrelation function need not vanish at different times.

It is, however, only after using the discretized stochastic process that we are able to validate our model. After correcting for the variability of the volatility by using the standardized daily yield, we have shown that for all 24 stocks the distribution of standardized daily yields is well described by the general Rademacher distribution. Indeed, we found that the abnormally large kurtosis is due to a volatility that changes with time. For 20 of the 24 stocks, the sample skewness, kurtosis, and probability distribution agrees with a

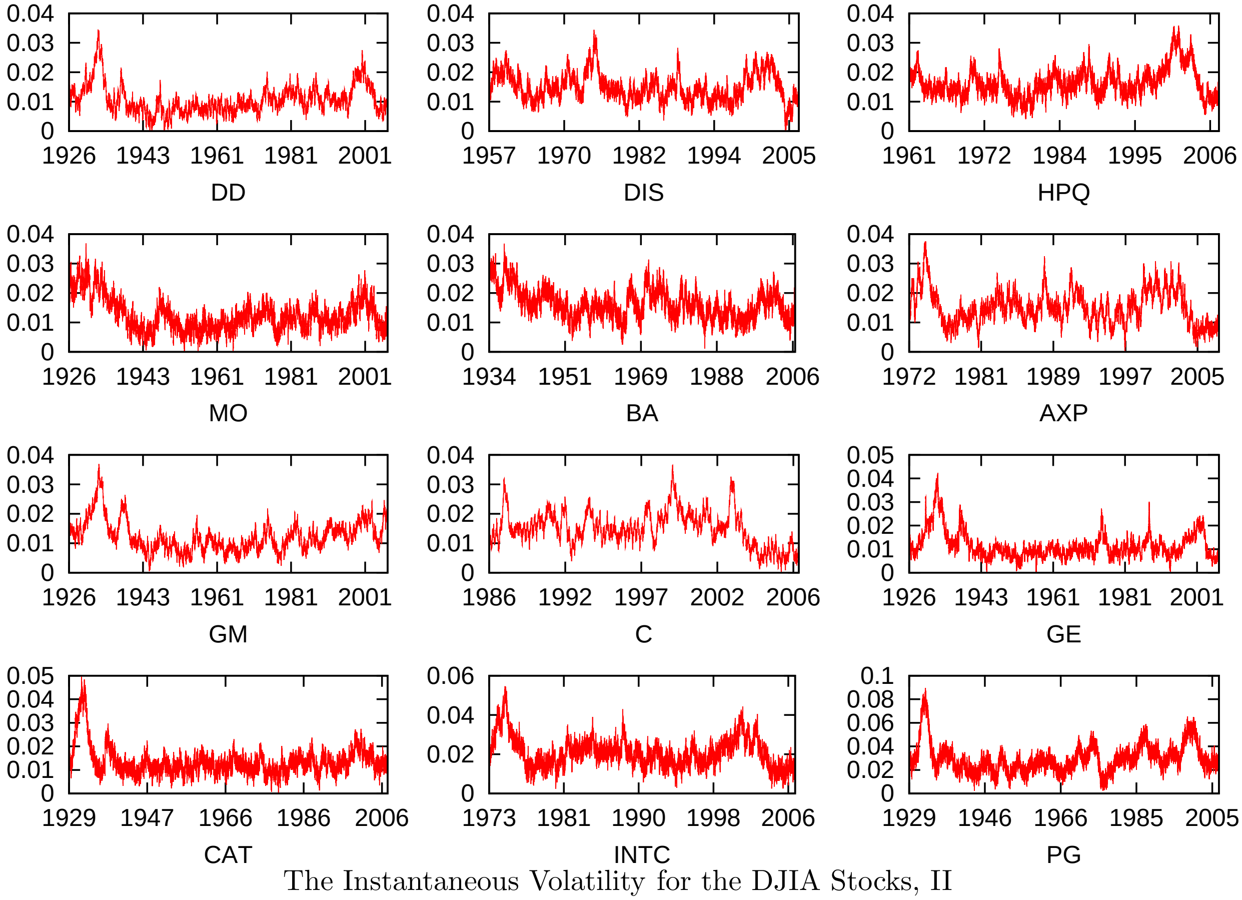


FIG. 6: Graphs of the instantaneous volatilities verses trading day for the 12 of the 24 DJIA stocks with the highest peak volatility are shown.

Rademacher distribution where $p = 1/2$ at the 95% CL; the probability that these stocks will increase on any one day is thus equal to the probability that it will decrease. The other four stocks agree with a generalized Rademacher distribution and have a p slightly greater than $1/2$. For these stocks, the probability that the yield will increase on any one day is slightly higher than the probability that it will decrease. We conclude that our model is a very good description of the behavior of these stocks.

That the kurtosis for the standardized daily yield is smaller than the kurtosis for the daily yield is in agreement with the results found by Rosenberg (1972). The daily yield is time dependent, and is thus a nonstationary random variable, while for the standardized daily yield, the time dependence due to the volatility has been taken account of. Indeed, in

many ways Rosenberg (1972) presages the results of this work.

By combining the properties of our continuous stochastic process for the stocks with noise removal techniques, we have been able to determine the time dependence of both the volatility and the drift of all 24 stocks. Unlike the implied volatility, the volatility obtained here was obtained from the daily close directly without the need to fit parameters to the market price of options. The theory is thus self-contained. For Alcoa, Caterpillar, and Johnson&Johnson, the time dependence of the volatility can be determined down to a resolution of a single trading day, while for another 13 stocks, they can be determined to a resolution of less than 1 1/2 trading days. While other, more sophisticated signal analysis techniques can be used, given that the time-series is based on the daily close and thus the resolution is ultimately limited to a period of two trading days, we do not expect that it will be possible to dramatically improve on these results. Only when intraday price data is used will we expect significant improvement to this resolution. Indeed, with intraday data we expect that changes to the volatility that occur during the trading day can be seen.

We have deliberately used large cap stocks in our analysis, and we take care to note that this approach to the analysis of the temporal behavior of stocks have only been shown to be valid for the 24 stocks we analyzed here. While we would expect it to be applicable to other large-cap stocks, whether our approach will also be valid when applied to mid- or small-cap stocks is still an open question. Indeed, it will be interesting to see the range of stocks for which the volatility depends solely on time.

With both the drift and the volatility determined down nearly to the single trading day level for most of the stocks, it is now be possible to calculate the autocorrelation function for both, as will as the correlation function between the drift and the volatility. In particular, the degree of influence that the volatility or drift on any one day has on the volatility or drift on any future day can be determined. This analysis is currently being done.

Appendix A: Preparing the Time-series

The time-series for the 24 DJIA stocks analyzed here were obtained from the Center for Research in Stock Prices (CRSP). While the ending date for each series is December 29, 2006, the choice of the starting date is often different for different stocks. This choice of starting dates was not governed by a desire for uniformity, but rather by the desire to include

as many trading days in the time series as possible, and thereby minimize standard errors. In addition, by maximizing the number of trading days included, we also demonstrate that our model is valid over the entire period for which the prices of the stock are available.

Although the daily close of stocks between the starting and ending dates are used as the basis of the time-series (with dividends included in the price), a series of adjustments to the CRSP data were made when the series were constructed. If the closing price of stock is listed by CRSP as a negative number—an indication that the closing price was not available on that day, and the average of the last bid and ask prices was used instead—we took the positive value of this number as the daily close on that day. If no record of the daily close was given for a particular trading day at all—an indication that the bidding and asking prices were also not available for that date—we used the average of the closing price of the stock on the day preceding and the day following as the daily close for that day.

Next, the daily close of the stock prices were scaled to adjust for splits in the stock. For example, although the daily close for Coca Cola on December 12, 1925 is listed by CRSP as \$153.625, this price was scaled by a factor of 6745.134 to account for the accumulated splits that the stock has gone through since 1925. The price recorded in the time-series is instead 0.022776. Because of this scale factor, the prices of stocks are listed in all time-series to an accuracy of at least 10^{-6} to ensure that the daily close on any day can be reconstructed from the time-series. This level of accuracy or higher was then used in all the calculations in this paper. While we could have avoided this subtlety by scaling the daily close, \$48.25, of the stock on December, 29, 2006 by 6745.134, doing so would result in stock prices that are ~\$300K, which is deceivingly large.

Finally, from Eq. (16) we see that σ_n vanishes if Δu_n vanishes, and yet from Eq. (15), it is implicit that the quotient $\Delta u_n/\sigma_n$ must be well defined. Indeed, the reduction of Eq. (3) to Eq. (7) is only valid if $\sigma(t)$ is nowhere zero. In practice, there are trading days on which the daily yield vanishes; for Coca Cola, this occurred 2070 out of a total of 21,523 trading days. To ensure that $\Delta u_n/\sigma_n$ is well defined on these days, we have added to the daily close a random number less than 0.00005 if the close on successive days are equal. This is done before the daily close is scaled to adjust for stock splits. Since the stock price changes by at least \$0.01 increments, doing so does not materially change the stock price, while still insuring that $\Delta u_n \neq 0$.

We have not adjusted for inflation in our time-series, nor have we accounted for week-

ends, holidays, or any other days on which trading did not take place. We have instead concatenated the daily close on each trading day, one after another, when constructing the time-series. The time-series are thus a sequence of *trading* days, and not calendar days. While this concatenation is natural, issues of bias such as those studied by Fleming, Kirby, and Ostdiek (2006) have not been taken into account. Whether these issues are relevant for the stocks considered here we leave for further study. Our focus is instead on the gross features of the stock price.

Finally, we list here the following particularities that occurred in our analysis of the 24 DJIA stocks.

Boeing: When solving for σ_n, the $n = 2$ term was greater than 27, while all other terms was less than 0.5. This data point was an outlier, and since it is in the transient region for the stock, we have set this term equal to 0.1, which is the typical size of σ_n for $n \neq 2$.

Merck: When solving for σ_n, the $n = 2$ term was greater than 168, while all other terms were three orders of magnitude smaller. This data point was replaced by 0.2, which is the typical size of σ_n for $n \neq 2$.

Exxon-Mobil: When solving for σ_n, the $n = 2$ term was greater than 158, while the $n = 3$ term was greater than 1500. The $n = 2$ data point was replaced by 0.1, and the $n = 3$ data point was replaced by 0.009.

Appendix B: Statistics

In this section, we collect the expressions used here in calculating the mean, variance, skewness, kurtosis, and autocorrelation of the time-series, along with their respective standard errors. With the exception of the autocorrelation function, these expressions are taken from Stuart and Ord (1994).

1. Moments and Standard Errors

Given a collection of N data points, x_n, the sample moments, m_k, of order, k, that are used in our analysis are defined as follows

$$
\begin{aligned}
m'_1 &\equiv \frac{1}{N}\sum_{n=1}^{N} x_n, \\
m_2 &\equiv \frac{1}{N-1}\sum_{n=1}^{N}(x_n - m'_1)^2, \\
m_3 &\equiv \frac{1}{(N-1)(N-2)}\sum_{n=1}^{N}(x_n - m'_1)^3, \\
m_4 &\equiv \frac{N(N+1)}{(N-1)(N-2)(N-3)}\sum_{n=1}^{N}(x_n - m'_1)^4, \\
m_5 &\equiv \frac{N^2(N+5)}{(N-1)(N-2)(N-3)(N-4)}\sum_{n=1}^{N}(x_n - m'_1)^5, \\
m_6 &\equiv \frac{N(N+1)(N^2+15N-4)}{(N-1)(N-2)(N-3)(N-4)(N-5)}\sum_{n=1}^{N}(x_n - m'_1)^6, \\
m_8 &\equiv \frac{N(N^5+99N^4+757N^3+114N^2-398N+120)}{(N-1)(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)}\sum_{n=1}^{N}(x_n - m'_1)^8, \quad \text{(B1)}
\end{aligned}
$$

As usual, the sample skewness and kurtosis are defined as

$$
skew = \frac{m_3}{m_2^{3/2}}, \qquad kurt = \frac{m_4}{m_2^2}. \qquad \text{(B2)}
$$

While the standard error of the mean and the variance is well known,

$$
\delta m'_1 = \sqrt{\frac{m_2}{N}}, \qquad \delta m_2 = \sqrt{\frac{m_4 - m_2^2}{N}}, \qquad \text{(B3)}
$$

the standard error in the sample skewness and kurtosis are not. For the skewness, this error is

$$
\delta\, skew = \frac{1}{\sqrt{N}}\left\{\frac{m_6}{m_2^3} - 6\frac{m_4}{m_2^2} + 9 + \frac{1}{4}\frac{m_3^2}{m_2^3}\left(9\frac{m_4}{m_2^2} + 35\right) - \frac{3m_5m_3}{m_2^4}\right\}^{1/2}, \qquad \text{(B4)}
$$

while for the sample kurtosis, the standard error is

$$
\delta\, kurt = \frac{1}{\sqrt{N}}\left\{\frac{m_8}{m_2^4} - 4\frac{m_6m_4}{m_2^5} + 4\left(\frac{m_4}{m_2^2}\right)^3 - \left(\frac{m_4}{m_2^2}\right)^2 + 16\frac{m_4m_3^2}{m_2^5} - 8\frac{m_5m_3}{m_2^4} + 16\frac{m_3^2}{m_2^3}\right\}^{1/2} \text{(B5)}
$$

Although standard errors are defined in terms of the population moments, these moments are not known a priorí. Following Stuart and Ord (1994), we have used instead the sample moments listed in Eq. (B1) when calculating standard errors.

2. The Autocorrelation Function

For the time-series, x_n, where $n = 1, \ldots, N$, we define the autocorrelation of x_N to be

$$G^{(2)}(x_N, x_{N-M}) \equiv \frac{1}{N-M} \sum_{i=M+1}^{N} \left(x_i - \frac{1}{N-M} \sum_{k=M+1}^{N} x_k \right) \left(x_{i-M} - \frac{1}{N-M} \sum_{k=M+1}^{N} x_{k-M} \right). \tag{B6}$$

Equation (B6) measures the correlation of the time-series at time step N with the time-series at time step $N-M$. This definition differs somewhat from the one given in Kendall (1953) and in Kendall, Stuart and Ord (1983) in that they divide $G^{(2)}(x_N, x_{N-M})$ by the product of the volatility of the time series, $\{x_n : n = M+1, \ldots, N\}$ with the volatility of of the time-series, $\{x_{n-M} : n = M+1, \ldots, N\}$. It also differs substantially from the expression used in Alexander (2001), where a simplified expression for the autocorrelation function in Kendall, Stuart, and Ord (1983) is used.

We use Eq. (B6) instead of the expressions given in Kendall, Stuart, and Ord (1983) and Alexander (2001) for two reasons. First, $G^{(2)}(x_N, x_N)$ is simply the variance of the time-series, so that the volatility for a stock can be read off easily from its graph, as can be seen in Fig. 1. Second, we will see below that the variance for $G^{(2)}(x_N, x_{N-M})$ is easily calculated when x_n are Gaussian random variables, and the standard error for $G^2(x_N, x_{N-M})$ can be readily determined. Derivations of the standard error for the autocorrelation functions given in Kendall, Stuart, and Ord (1983) and Alexander (2001), on the other hand, are more involved.

To determine the standard error for $G^{(2)}(x_N, x_{N-M})$, consider a time-series where the x_n are Gaussian random variables with mean zero and standard deviation, σ^2. Then $E[x_i] = 0$ while $E[x_i x_k] = \sigma^2 \delta_{ik}$. (Here, δ_{ik} is the Kronecker delta with $\delta_{ik} = 1$ if $i = k$ while $\delta_{ik} = 0$ otherwise.) Consequently, $E\left[G^{(2)}(x_n, x_{n-M})\right] = 0$ when $M > 0$, as can be seen from Eq. (B6). We thus only have to calculate

$$E\left[\left\{G^{(2)}(x_N, x_{N-M})\right\}^2\right] = \frac{1}{(N-M)^2} \sum_{i=M+1}^{N} \sum_{k=M+1}^{N} E[x_i x_{i-M} x_k x_{k-M}]. \tag{B7}$$

Since the x_n are Gaussian random variables,

$$E[x_i x_j x_k x_l] = \sigma^4 \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right). \tag{B8}$$

Thus,

$$E\left[\left\{G^{(2)}(x_N, x_{N-M})\right\}^2\right] \;=\; \frac{\sigma^4}{(N-M)^2}\sum_{i,k=M+1}^{N}\Big(\delta_{i,i-M}\delta_{k,k-M} + \delta_{ik}\delta_{i-M,k-M} + \delta_{i,k-M}\delta_{i-M,k}\Big). \tag{B9}$$

The first term vanishes since $M > 0$, while the third term vanishes because it requires that $i = k - M$ and $i - M = k$; this can only happen when $M = 0$. We are thus left with only the second term, so that

$$E\left[\left\{G^{(2)}(x_N, x_{N-M})\right\}^2\right] = \frac{\sigma^4}{N-M}. \tag{B10}$$

The standard error, $\Delta G^{(2)}(x_N, x_{N-M})$, for $G^{(2)}(x_N, x_{N-M})$ when $M > 0$ is then simply

$$\Delta G^{(2)}(x_N, x_{N-M}) = \frac{G^{(2)}(x_N, x_N)}{\sqrt{N-M}}, \tag{B11}$$

where we have used the fact that $G^{(2)}(x_N, x_N)$ is the variance of the time-series. The standard error for $G^{(2)}(x_N, x_{N-M})$ when $M = 0$ can then be found from Eq. (B3) after remembering that $m_4 = 3\sigma^4$ for a Gaussian distribution. Note that differences between Eq. (B11) and the standard error found in Kendall, Stuart, and Ord (1983) are due mainly to our defining $G^{(2)}(x_N, x_{N-M})$ with the factor $1/(N-M)$ instead of the factor $1/N$ used by them.

Although the standard error for $G^{(2)}(x_N, x_{N-M})$ when x_n is not a Gaussian random variable can be found for special cases (see Kendall, Stuart, and Ord 1983), Eq. (B11) is sufficient for our purposes. If the market is efficient, we expect the autocorrelation function for the standard yield to vanish for $M > 0$. As this expectation is borne out by Fig. 1, we hypothesize that the reason why the autocorrelation function in Fig. 1 is not identically zero when $T > 0$ is due to sample errors, which in turn is due to Gaussian random variables with zero mean. We would therefore expect Eq. (B11) to be a good description of the standard error of this autocorrelation function, and indeed, this expectation is consistent with the results listed in Table I.

Appendix C: Fourier Analysis

In this appendix, we review the properties of the Fourier transform needed in the analysis we present in this paper. While much of this is well-known, our purpose here is to establish

the notation used in the paper, and to review the properties of Fourier series needed. At the end of this section, we will also show that the Fourier transform of a Gaussian random variable is once again a Gaussian random variable, and describe the method used to remove the noise from σ_n.

1. The Discrete Fourier Transform

Consider a times-series, x_n, such that $n = N_{min}, \ldots, N_{max}$; the total number of data points in the time-series is then $N = N_{max} - N_{min} + 1$. In the analysis below, we will assume that N is an odd number. As there are at least 5,000 trading days in our time-series, we can always change the starting point of a time-series by one trading day to insure that there are an odd number of terms in the series; this assumption is thus not restrictive.

The expansion of the time-series in a Fourier series is defined as

$$x_n = \sum_{k=-(N-1)/2}^{(N-1)/2} x_k^\omega e^{-2\pi ikn/N}, \tag{C1}$$

where $i = \sqrt{-1}$. The quantity, x_k^ω, is called the Fourier transform of x_n. As

$$\exp\left(-\frac{2\pi ikn}{N}\right) = \cos\left(\frac{2\pi kn}{N}\right) - i\sin\left(\frac{2\pi kn}{N}\right), \tag{C2}$$

in taking the Fourier series of x_n we have decomposed x_n into terms that oscillate with definite period, $T_k = N/k$, for $k > 0$, and have a definite amplitude, $|x_k^\omega|$. This transform is thus a natural method of characterizing how a time-series changes with time.

The amplitude of these oscillations, x_k^ω, is a complex number in general. The original time-series, x_n, is real, however, and this fact must also be reflected in x_k^ω. How it is reflected can be seen by taking the complex conjugate of Eq. (C1),

$$\bar{x}_n = \sum_{k=-(N-1)/2}^{(N-1)/2} \bar{x}_k^\omega e^{2\pi ikn/N}, \tag{C3}$$

where the complex conjugate is denoted by a bar. Since $x_n = \bar{x}_n$, by comparing Eq. (C1) with Eq. (C3) we find after taking $k \to -k$ in Eq. (C1) the reality condition $\bar{x}_k^\omega = x_{-k}^\omega$ that the Fourier transform must satisfy.

The transform Eq. (C1) is invertible. Namely, we can express x_k^ω in terms of x_n by taking

the following sum

$$\sum_{n=N_{min}}^{N_{max}} x_n \exp\left(\frac{2\pi\hat{k}n}{N}\right) = \sum_{k=-(N-1)/2}^{(N-1)/2} x_k^\omega \sum_{n=N_{min}}^{N_{max}} \left[\exp\left(\frac{2\pi i(\hat{k}-k)}{N}\right)\right]^n, \tag{C4}$$

of Eq. (C1). To evaluate this sum, we consider first the case where $\hat{k}-k \neq qN$ for any integer q. The series on the right can then be summed to give

$$\sum_{n=N_{min}}^{N_{max}} \left(e^{2\pi i(\hat{k}-k)/N}\right)^n = e^{\left[2\pi i(\hat{k}-k)N_{min}/N\right]} \left(\frac{1-e^{2\pi i(\hat{k}-k)}}{1-e^{2\pi i(\hat{k}-k)/N}}\right), \tag{C5}$$

after using the following identity for the geometric series,

$$\sum_{n=0}^{N} y^n = \frac{1-y^{N+1}}{1-y}. \tag{C6}$$

As $e^{2\pi i(\hat{k}-k)} = 1$, while $e^{2\pi i(\hat{k}-k)/N} \neq 1$, we conclude that Eq. (C5) vanishes in this case. We next consider the case when $\hat{k}-k = qN$. Each term in the sum is then one, and Eq. (C5) is easily summed to give N.

Combining these two results, we find that

$$\sum_{n=N_{min}}^{N_{max}} \left(e^{2\pi i(\hat{k}-k)/N}\right)^n = N\delta_{\hat{k},k}. \tag{C7}$$

We then conclude from Eq. (C4) that

$$x_k^\omega = \frac{1}{N} \sum_{n=N_{min}}^{N_{max}} x_n e^{2\pi i k n/N}. \tag{C8}$$

This is the inverse Fourier transform of x_n. In particular, notice that when $k=0$,

$$x_0^\omega = \frac{1}{N} \sum_{n=N_{min}}^{N_{max}} x_n, \tag{C9}$$

is simply the average of x_n over the whole time-series.

The Fourier series Eq. (C1) can also be expressed as an explicitly real expansion,

$$x_n = a_0^{\cos} + 2\sum_{k=1}^{(N-1)/2} a_k^{\cos} \cos\left(\frac{2\pi kn}{N}\right) + 2\sum_{k=1}^{(N-1)/2} a_k^{\sin} \sin\left(\frac{2\pi kn}{N}\right), \tag{C10}$$

where the amplitudes

$$\begin{aligned} a_k^{\cos} &\equiv \frac{1}{2}\left(x_k^\omega + \bar{x}_k^\omega\right) = \frac{1}{N}\sum_{n=N_{min}}^{N_{max}} x_n \cos(2\pi nk/N), \\ a_k^{\sin} &\equiv \frac{1}{2i}\left(x_k^\omega - \bar{x}_k^\omega\right) = \frac{1}{N}\sum_{n=N_{min}}^{N_{max}} x_n \sin(2\pi nk/N). \end{aligned} \tag{C11}$$

are called the Fourier cosine and Fourier sine coefficients, respectively. While analytical calculations are more easily done with Eq. (C1), numerical calculations are necessarily done with Eq. (C10), and it is on the Fourier sine and cosine coefficients that we will focus most of our analysis in this paper.

2. Fourier Transforms of Gaussian Random Variables

We now prove the theorem stated in Sec. VI for the special case when N is an odd number. Although the theorem holds in general, this is the only case we need here.

Because each ξ_n in the time-series is a Gaussian random variable with zero mean and variance, σ^2, the probability distribution for the time-series is just

$$P(\xi_1,\ldots,\xi_N) = \frac{1}{\sigma^N}\left(\frac{a}{2\pi}\right)^{N/2}\prod_{n=1}^{N} e^{-\xi_n^2/2\sigma^2} = \frac{1}{\sigma^N}\left(\frac{a}{2\pi}\right)^{N/2}\exp\left(-\frac{1}{2\sigma^2}\sum_{n=1}^{N}\xi_n^2\right), \qquad \text{(C12)}$$

where a is the time interval between successive points in the time series, and in the last equality we have used $E[\xi_n\xi_m] = 0$ for $n \neq m$. Expanding ξ_n in a Fourier series using Eq. (C1), we find that

$$\sum_{n=1}^{N}\xi_n^2 = \sum_{k,k'=-(N-1)/2}^{(N-1)/2}\xi_k^\omega\xi_{k'}^\omega\sum_{n=1}^{N}\exp\left[-\frac{2\pi i n}{N}(k+k')\right] = \sum_{k,k'=-(N-1)/2}^{(N-1)/2}\xi_k^\omega\xi_{k'}^\omega N\delta_{k,-k'}, \qquad \text{(C13)}$$

where the last equality holds from Eq. (C7). Thus,

$$\sum_{n=1}^{N}\xi_n^2 = N\sum_{k=-(N-1)/2}^{(N-1)/2}|\xi_k^\omega|^2, \qquad \text{(C14)}$$

which is Parseval's Theorem for a discrete Fourier series. Following Eq. (C11), we express $|\xi_k^\omega|^2 = \left(\alpha_k^{\sin}\right)^2 + (\alpha_k^{\cos})^2$ in Eq. (C14). Then Eq. (C12) can be written as

$$P = \frac{1}{\sigma^N}\left(\frac{a}{2\pi}\right)^{N/2}\prod_{k=-(N-1)/2}^{(N-1)/2}\exp\left\{-\frac{N}{2\sigma^2}\left[\left(\alpha_k^{\sin}\right)^2 + (\alpha_k^{\cos})^2\right]\right\}, \qquad \text{(C15)}$$

and the theorem is proved.

It is straightforward to see that the converse is also true. Namely, if $\alpha_k^{\cos}$ and $\alpha_k^{\sin}$ are Gaussian random variables with zero mean and volatility $\sigma/\sqrt{N}$, then x_n is a Gaussian random variable with zero mean and volatility, σ.

Notice from Eq. (C15) that $E[\alpha_k^{\cos}\alpha_k^{\sin}] = 0$, and thus the two random variables are independent. Notice also that while we began with N degrees of freedom with the random variables, x_n, we seem to have ended up with $2N-1$ degrees of freedom for the random variables, $\alpha_k^{\cos}$ and α_k^{sin}. From Eq. (C11) we see, however, that $\alpha_k^{\cos} = \alpha_{-k}^{\cos}$ and $\alpha_k^{\sin} = \alpha_{-k}^{\sin}$; not all the variables in Eq. (C15) are independent. When this redundancy is taken account of, we arrive back to N degrees of freedom.

3. Removal of Noise

In this subsection, we will describe how the noise present in σ_n is removed, and how $\sigma(t)$ is obtained.

Given that the noise floor associated with the Fourier sine and cosine coefficients is constant over all frequencies, f_k, noise removal is straight forward. We need only remove from $\mathcal{F}^{\sin}$, the set of all Fourier sine coefficients for σ_n, and $\mathcal{F}^{\cos}$, the set of all Fourier cosine coefficients for σ_n, those coefficients whose amplitudes is less than the amplitudes, S_{noise} and C_{noise}, of the noise floor for the Fourier sine and Fourier cosine coefficients, respectively. The coefficients left over—$\mathcal{F}_{signal}^{\sin} = \{a_k^{\sin} \in \mathcal{F}^{\sin} : |a_k^{\sin}| > S_{noise}\}$ for the Fourier sine coefficients, and $\mathcal{F}_{signal}^{\cos} = \{a_k^{\cos} \in \mathcal{F}^{\cos} : |a_k^{\cos}| > C_{noise}\}$ for the Fourier cosine coefficients—can then be used to construct the instantaneous volatility, $\sigma(t)$, by summing the Fourier series Eq. (29).

The noise floor amplitudes, S_{noise} and C_{noise}, are determined statistically. Consider the set of coefficients that are removed: $\mathcal{F}_{noise}^{\sin} = \{a_k^{\sin} \in \mathcal{F}^{\sin} : |a_k^{\sin}| \leq S_{noise}\}$ for the Fourier sine coefficients and $\mathcal{F}_{noise}^{\cos} = \{a_k^{\cos} \in \mathcal{F}^{\cos} : |a_k^{\cos}| \leq C_{noise}\}$ for the Fourier cosine coefficients. Because the distribution of the noise floor is Gaussian, S_{noise} and C_{noise} must be chosen so that the distributions of coefficients in $\mathcal{F}_{noise}^{\sin}$ and $\mathcal{F}_{noise}^{\cos}$ are Gaussian as well. If either amplitude is chosen too large, then coefficients from $\mathcal{F}^{\sin}$ or $\mathcal{F}^{\cos}$ that make up the signal, $\sigma(t)$, would be included in the noise distributions as noise. As these coefficients are supposed to be above the noise, they will skew and flatten the distribution; the skewness and the kurtosis for the distribution of $\mathcal{F}_{noise}^{\sin}$ and of $\mathcal{F}_{noise}^{\cos}$ will then differ from their Gaussian values if these coefficients are included. On the other hand, if either amplitude for the noise floor is chosen too *small*, then coefficients from $\mathcal{F}^{\sin}$ or $\mathcal{F}^{\cos}$ that make up the noise would be *excluded* from the noise distributions. As these coefficients would have populated the tails of the Gaussian distribution, their removal will tend to *narrow* the distribution,

and the kurtosis of the noise distributions will differ once again from its Gaussian value. (Because the coefficients are remove symmetrically about the horizontal zero line, a choice of the amplitude for the noise floor that is too small will not tend to change the skewness significantly.) Thus, S_{noise} and C_{noise} must be chosen so that the skewness and kurtosis of the distribution of coefficients in $\mathcal{F}^{\sin}_{noise}$ and $\mathcal{F}^{\cos}_{noise}$ is as close to their Gaussian distribution values as possible.

While the above procedure is straightforward, there is an additional constraint. The volatility cannot be negative, and thus the resultant instantaneous volatility, $\sigma(t)$, obtained after the noise floor is removed must the positive as well. This constraint is not trivial. For a number of stocks, a choice of S_{noise} and C_{noise} that results in noise distributions that are closest to a Gaussian distribution also results in a $\sigma(t)$ that is negative on certain days. To obtain a $\sigma(t)$ that is non-negative, slightly larger amplitudes for the noise floors were chosen, which resulted in a slightly larger skewness and kurtosis.

This approach to removing the noise from the volatility, σ_n, has been successfully applied to all 24 stocks using a simple C++ program that implements an iterative search algorithm to determine S_{noise} and C_{noise}. The results of our numerical analysis are shown in Table IV. There, we have listed the noise floor amplitudes, S_{noise} and C_{noise}, used for each of the 24 stocks. Their values are given as multiples of the standard deviation of the distribution of the Fourier coefficients in $\mathcal{F}^{\sin}_{noise}$ and $\mathcal{F}^{\cos}_{noise}$. As these values range from 3.030 times the standard deviation to 3.948 times the standard deviation, 99.756% to 99.992% of the data points that make up a Gaussian distribution can be included in these distributions if they are present in either $\mathcal{F}^{\sin}_{noise}$ or $\mathcal{F}^{\cos}_{noise}$.

Listed also in Table IV are the kurtosis for $\mathcal{F}^{\sin}_{noise}$ and $\mathcal{F}^{\cos}_{noise}$. We have found that they range in value from 2.95 to 3.31, and are thus very close to the Gaussian distribution value of three for the kurtosis. The skewness of the noise of the distribution of $\mathcal{F}^{\cos}_{noise}$ was calculated as well, and was found to vary in value from -0.03 to 0.11; this also is very close to the Gaussian distribution value of zero for the skewness. The skewness for the distribution of $\mathcal{F}^{\sin}_{noise}$ was also calculated, but we find that their values are 10^{-2} to 10^{-7} times smaller than the skewness for the distribution of $\mathcal{F}^{\cos}_{noise}$, and there was no need to listed these values in the table. This extremely close agreement with the skewness of the Gaussian distribution is because the Fourier sine coefficients are antisymmetric about $k = 0$: $a^{\sin}_k = -a^{\sin}_{-k}$. The average of any odd power of $a^{\sin}_k$ over k—and in particular, the skewness of the distribution

of $\mathcal{F}^{\sin}$—thus automatically vanishes. For this reason, the skewness for $\mathcal{F}^{\sin}_{noise}$ is exceedingly small.

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